1. Suppose \( V \) is a vector space over \( \mathbb{F} \) with \( \dim V = n \). Suppose \( T \in L(V, V) \) is an operator on \( V \). Suppose \( W_0 \) is an \( T \)-invariant subspace of \( V \) and \( v \in V \setminus W_0 \). Write \( W_1 = W_0 + \mathbb{F}[T]v \) and \[ I = \{ f \in \mathbb{F}[X] : f(T)v \in W_0 \}. \]

Prove that

(a) \( I \) is a proper ideal,
(b) Let \( f \) be the MMP of \( I \). Prove that \( \dim W_1 = \dim W_0 + \text{degree}(f) \).

2. Suppose \( V \) is a vector space over \( \mathbb{F} \) with \( \dim V = n \). Suppose \( T \in L(V, V) \) is an operator on \( V \). Prove that \( V \) is cyclic if and only if \( V \) has a basis \( E \) such that with respect to \( E \) the matrix of \( T \) is the companion matrix of a monic polynomial \( p \).

3. Let \( p \) be a non-constant monic polynomial and \( A \) be the companion matrix of \( p \). Prove that \( p \) is both the MMP of \( A \) and the characteristic polynomial of \( A \).

4. Let \( T : \mathbb{F}^3 \rightarrow \mathbb{F}^3 \) be the operator defined by

\[
T(X) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} X
\]

Prove that \( \mathbb{F}^3 \) is not \( T \)-cyclic.

5. Suppose \( V \) is a vector space over \( \mathbb{F} \) with \( \dim V = n \). Suppose \( T \in L(V, V) \) is an operator on \( V \). Suppose \( w_1, \ldots, w_r \) be such that

(a) \( V = \mathbb{F}[T]w_1 \oplus \mathbb{F}[T]w_2 \oplus \cdots \oplus \mathbb{F}[T]w_r \).

(b) Let \( p_i \) be the MMP of \( w_i \). Assume that \( p_k | p_{k-1} \) for \( k = 2, \ldots, r \).

Prove \( \text{ann}(T) = \text{ann}(w_1) = \mathbb{F}[X]p_1 \).
6. Suppose $V$ is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on $V$. Assume $W_0$ is a $T$–admissible set and $V = W_0 + \mathbb{F}[T]v$ for some $v \in V$. Find $w \in V$ such that

(a) $V = W_0 \oplus \mathbb{F}[T]w$.

(b) Let $p$ be the MMP of $w$. Prove that $p$ is unique. That means, if $V = W_0 \oplus \mathbb{F}[T]w'$ for some $w'$ then MMP of $w'$ is $p$.

7. Suppose $V$ is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on $V$. Then $V$ is $T$–cyclic if and only if the characteristic polynomial and the MMP of $T$ are identical.

8. Suppose $V$ is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on $V$. Let $P$ be the MMP of $T$ and $Q$ be the characteristic polynomial of $T$. Let $p$ be an irreducible polynomial. Prove that

$$p \mid P \iff p \mid Q.$$ 

9. Given an example (with justification) of a $2 \times 2$–matrix $A$ whose characteristic polynomial is $(1 - X)^2$.

10. Suppose $V$ is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V,V)$ is an operator on $V$. Suppose the characteristic polynomial $Q$ of $T$ factors completely into linear factors:

$$Q = (X - c_1)^{d_1}(X - c_2)^{d_2}(X - c_k)^{d_k}$$

where $c_1, \ldots, c_k$ are the distinct eigen values of $T$. Describe the Jordan matrix of $T$ and give a short outline of the proof of existence.