1. Suppose $V$ is vector space over $\mathbb{F}$ with finite dim$(V) = n$. Let $T \in L(V, V)$ be a linear operator. Suppose $W$ is a $T$–invariant subspace of $W$ and $T' = T|_W$ be the restriction.

(a) Let $q$ be the characteristic polynomial of $T$ and $Q$ be the characteristic polynomial of $T'$. Prove that $Q \mid q$.

(b) Likewise, let $p$ bet the MMP of $T$ and $P$ be the MMP of $T'$. Prove that $P \mid p$.

2. Suppose $V$ is vector space over $\mathbb{F}$ with finite dim$(V) = n$. Let $T \in L(V, V)$ be a linear operator. Suppose there is a basis $E$ of $V$ such that the matrix $A$ of $T$ with respect to $E$ is upper triangular. Prove that there is another basis $E'$, such that with respect to $E'$, the matrix of $T$ is lower triangular.

3. Let $V$ be a vector space over $\mathbb{F}$ with with finite dimension dim $V = 3$ and $T : V \rightarrow V$ be a linear operator on $V$. Prove that $T$ is triangulable if and only if the minimal polynomial $p$ of $T$ is a product of linear factors.

4. Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$.

(a) Let $e_1, e_2, \ldots, e_k$ be elements of $V$. Prove that $e_1, e_2, \ldots, e_k$ are linearly independent if and only if $j = 1, \ldots, k$, we have $e_j \notin \text{Span}(e_1, e_2, \ldots, e_{j-1})$.

(b) Let $W_1, \ldots, W_k$ be subspaces of $V$. Prove that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ if and only if $V = W_1 + W_2 + \cdots + W_k$ and for each $j = 2, \ldots, k$, we have

$$(W_1 + \cdots + W_{j-1}) \cap W_j = \{0\}.$$
(a) $E_i E_j = 0 \quad \forall \quad i \neq j.$
(b) $E_1 + E_2 + \cdots + E_k = I.$

Write $W_i = E_i(V)$. Prove that $E_i$ is a projection and

\[ V = W_1 \oplus W_2 \oplus \cdots \oplus W_k. \]

6. Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$ and $W$ be a subspace of $V$. Prove that there is subspace $U$ of $V$ such that $V = W \oplus U$.

7. Let $V = \mathbb{R}^2$.
   (a) Write down the the projection $\pi : V \to V$ to the line $y = x$.
   (b) Let $\mathbf{e} = (0, 1) \in V$. Write down the projection $p : V \setminus \{P\} \to V$ to the $x$–axis from the point $P$. Is it linear?

8. Let $V$ be a vector space over $\mathbb{F}$ with finite dimension $\dim V = n$ and $T : V \to V$ be a linear operator on $V$. Let $p$ be the minimal monic polynomial (MMP) of $T$ and

\[ p = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \]

where $r_i > 0$ and $p_i$ are distinct irreducible monic polynomials in $\mathbb{F}[X]$. Let

\[ W_i = \{v \in V : p_i(T)^{r_i}(v) = 0\} \]

be the null space of $p_i(T)^{r_i}$.

(a) Prove that $W_1$ is invariant under $T$.
(b) Let

\[ f_1 g_1 + f_2 g_2 + \cdots + f_k g_k = 1 \]

where $f_i = \prod_{j \neq i} p_j^{r_j}$. Prove that $W_1 = f_1 g_1(T)(V)$.
(c) Prove that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.
(d) Let $T_1 = T|_{W_1}$ be the restriction of $T$. Prove that MMP of $T_1$ is $p_1^{r_1}$. 

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