Unless otherwise stated, $\mathbb{F}$ is a field and $\mathbb{F}[X]$ is the polynomial ring over $\mathbb{F}$.

1. Prove that a polynomial $f(X) \in \mathbb{F}[X]$ of degree $n$ has atmost $n$ roots in $\mathbb{F}$.

2. Suppose $I$ is a non-zero ideal in $\mathbb{F}[X]$. Prove that $I = \mathbb{F}[X]d$ for some $d \in I$.

3. Suppose $f_1, \ldots, f_n \in \mathbb{F}[X]$ and not all of them are zero. Prove that they have a CGD and the two GCDs differ by a unit multiple.

4. Suppose $f_1, \ldots, f_n \in \mathbb{F}[X]$ are all non-zero. Suppose $p$ is a prime element. Prove that if $p$ divides the product $f_1 f_2 \cdots f_n$ then $p$ divides $f_i$ for some $i = 1, \ldots, n$.

5. Prove that any $f \in \mathbb{F}[X]$ has a unique factorization as

$$f = up_1 p_2 \cdots p_r$$

where $u$ is an unit and $p_i$ is a prime in $\mathbb{F}[X]$ for $i = 1, \ldots, r$.

6. (About determinants) Let $V$ be a vector space over $\mathbb{F}$ with $\text{dim}(V) = n$. Let $T \in L(V, V)$
be a linear operator. Define determinant of $T$ and prove that it is well defined.