1. Let $V$ be a vector space over $F$ and $W$ be a non-empty subset of $V$. Prove that the following are equivalent:

(a) $W$ is a subspace of $V$.
(b) For $u, v \in W$ and $c, d \in F$ we have $cu + dv \in W$.
(c) For $u, v \in W$ and $c \in F$ we have $u + v \in W$ and $cu \in W$.
(d) For $u, v \in W$ and $c \in F$ we have $cu + v \in W$.

2. Let $V$ be a vector space over $F$ and $S$ be a non-empty subset of $V$.

(a) Define the subspace spanned by $S$. Write $W = \text{Span}(S)$.
(b) Prove that if $U$ is a subspace of $V$ containing $S$, then $W$ is contained in $U$.
(c) Prove

$$W = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n : n \geq 0, c_i \in F, v_i \in S\}.$$
3. Let \( V \) be a vector space over \( F \) and \( V \) is spanned by a finite set \( S = \{v_1, \ldots, v_n\} \). Prove that a subset of \( S \) will form a basis of \( V \).

4. Let \( V \) be a finitely dimensional vector space over \( F \) let \( S = \{v_1, \ldots, v_n\} \) be a linearly independent subset. Prove that \( S \) extends to a basis of \( V \). \( (\text{We really do not need to assume that } V \text{ has finite dimension.}) \)

5. Let \( V \) be a vector space over \( F \) and \( V \) is spanned by a finite set \( S = \{v_1, \ldots, v_n\} \). Prove that any two basis of \( V \) have same number of elements. \( (\text{We really do not need to assume that } S \text{ is a finite set.}) \)

6. Let \( V \) be a vector space over \( F \) and \( W_1, W_2 \) be two subspaces of \( V \). Assume \( W_1 + W_2 \) has finite dimension. Prove that
\[
\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).
\]

7. Let \( A, B \) be two \( m \times n \) matrices with entries in \( F \). Prove that \( A \) and \( B \) have same row space if and only if they are row equivalent.

8. Let \( V = F[X] \) be set of all polynomials over \( F \). Prove that, as a vector space, \( V \) does not have finite dimension.