1. Let $V$ be a vector space over $\mathbb{F}$ and $W$ be a non-empty subset of $V$. Prove that the following are equivalent:

(a) $W$ is a subspace of $V$.
(b) For $u, v \in W$ and $c, d \in \mathbb{F}$ we have $cu + dv \in W$.
(c) For $u, v \in W$ and $c \in \mathbb{F}$ we have $u + v \in W$ and $cu \in W$.
(d) For $u, v \in W$ and $c \in \mathbb{F}$ we have $cu + v \in W$.

Solution. Similar to the proof of Problem-1 in Test-3.
2. Let $V$ be a vector space over $\mathbb{F}$ and $S$ be a non-empty subset of $V$.

(a) Define the subspace spanned by $S$. Write $W = \text{Span}(S)$.

(b) Prove that if $U$ is a subspace of $V$ containing $S$, then $W$ is contained in $U$.

(c) Prove

$$W = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n : n \geq 0, c_i \in \mathbb{F}, v_i \in S\}.$$ 

Solution. (a) We define $\text{Span}(S)$ to be the intersection of all subspaces $L$ of $V$ that contain $S$. Notationally,

$$W = \text{Span}(S) = \cap \{L : L \text{ subspace of } V, S \subseteq L\}.$$ 

Proof of (b) Suppose $U$ is a subspace of $V$ and $S \subseteq U$. Then, $U$ is a member of the family on the RHS of the definition. So, $W \subseteq U$.

Proof of (c) Write

$$L_0 = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n : n \geq 0, c_i \in \mathbb{F}, v_i \in S\}.$$ 

Now, $L_0$ is a subspace of $V$ and $S \subseteq L_0$. So, by (b), the span $W \subseteq L_0$.

Now suppose $v \in W$. Then $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ with $c_i \in \mathbb{F}$ and $v_i \in S$. If $L$ is a subspace of $V$ and $S \subseteq L$ then $v \in L$. Therefore, $v \in L$, for each member $L$ of the family of subspaces in the definition. So, $v \in \cap \{L : L \text{ subspace of } V, S \subseteq L\} = W$. Therefore $L_0 \subseteq W$. So, the proof is complete.
3. Let $V$ be a vector space over $\mathbb{F}$ and $V$ is spanned by a finite set $S = \{v_1, \ldots, v_n\}$. Prove that a subset of $S$ will form a basis of $V$.

**Solution.** If $V = \{0\}$, then empty set $\emptyset$ forms a basis and the statement hold. So, assume $V \neq \{0\}$.

Let $i_1 = \text{minimum}\{i : v_i \neq 0\}$. Write $W_1 = \text{Span}(v_{i_1})$. Then $v_{i_1}$ is linearly independent. If $W_1 = \text{Span}(v_{i_1}) = V$ then $v_{i_1}$ is a basis of $V$ and we are through.

So, we assume $W_1 \not\subseteq V$. Write $S_2 = \{v_i : i_1 < i \leq n\}$. Since $W_1 \not\subseteq V$ there are elements $v_i \in S_2 \setminus W_1$. Let $i_2 = \text{minimum}\{i : v_i \in S_2, v_i \notin W_1\}$.

Since $v_{i_2} \notin W_1$, we have $v_{i_1}, v_{i_2}$ are linearly independent.

Write $W_2 = \text{Span}(v_{i_1}, v_{i_2})$. If $W_2 = V$ then $v_{i_1}, v_{i_2}$ is a basis of $V$ and we are through.

So, we assume $W_2 \not\subseteq V$. Since $S$ is finite, this process must terminate and we will get a basis $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ of $V$. So, the proof is complete.

4. Let $V$ be a finite dimensional vector space over $\mathbb{F}$ let $S = \{v_1, \ldots, v_n\}$ be a linearly independent subset. Prove that $S$ extends to a basis of $V$. *(We really do not need to assume that $V$ has finite dimension.)*
5. Let $V$ be a vector space over $\mathbb{F}$ and $V$ is spanned by a finite set $S = \{v_1, \ldots, v_n\}$. Prove that any two basis of $V$ have same number of elements. \textit{(We really do not need to assume that $S$ is a finite set.)}

\textbf{Solution.} Since $V$ is spanned by a finite set, it has a finite basis. Suppose $e_1, \ldots, e_r$ be a basis of $V$ with $r$ elements and $E_1, \ldots, E_s$ be a basis of $V$ with $r$ elements.

Suppose $r \neq s$. Assume $s < r$. We have

$$(e_1, \ldots, e_r) = (E_1, \ldots, E_s)A$$

for some $s \times r$ matrix $A$.

Now the homogeneus system $AX = 0$ has $s$ equations in $r$ unknown. Since $s < r$, this system has a non-zero solution $C = (c_1, \ldots, c_r)^t$, where $c_i \in \mathbb{F}$. So, $AC^t = 0$. Therefore, $c_1e_1 + \cdots + c_re_r = 0$. This is contrdicts that $e_1, \ldots, e_r$ is a basis.

Therefore $r = s$ and the proof is complete.

6. Let $V$ be a vector space over $\mathbb{F}$ and $W_1, W_2$ be two subspaces of $V$. Assume $W_1 + W_2$ has finite dimension. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$
7. Let $A, B$ be two $m \times n$ matrices with entries in $\mathbb{F}$. Prove that $A$ and $B$ have same row space if and only if they are row equivalent.

8. Let $V = \mathbb{F}[X]$ be set of all polynomials over $\mathbb{F}$. Prove that, as a vector space, $V$ does not have finite dimension.

**Solution.** Suppose dim $V = n$ is finite and $f_1, f_2, \ldots, f_n$ be basis of $V$. Let

$$d = \text{maximum}\{\text{degree}(f_1), \text{degree}(f_2), \ldots, \text{degree}(f_n)\}.$$ 

Then

$$V = \text{Span}(f_1, f_2, \ldots, f_n) \subseteq \sum_{i=0}^{d} \mathbb{F}X^i.$$ 

This is a contradiction. Because a polynomial $f$ with $\text{degree}(f) > d$ is not in the sum on the right hand side. In particular, $X^{d+1} \notin \sum_{i=0}^{d} \mathbb{F}X^i$. So, the proof is complete.