I like short proofs and direct proofs.

1. Suppose $V$ is a vector space over $\mathbb{F}$ and $W \subseteq V$ is a subspace of $V$. Prove that annihilator of the the annihilator of $W$ is itself. That is, notationally, prove that $W = W^{00}$.

2. Suppose $V$ is vector space of finite dimension, $\dim V = n$, over $\mathbb{F}$. Let $g, f_1, \ldots, f_r \in V^*$ be linear functionals. Let $N$ be the null space of $g$ and $N_i$ be the null space of $f_i$.
   
   Then, $N_1 \cap N_2 \cap \cdots \cap N_r \subseteq N$ if and only if $g = \sum_{i=1}^r c_i f_i$ for some $c_i \in \mathbb{F}$.

3. Suppose $V$ is a vector space over $\mathbb{F}$ and $W \subseteq V$ is a subspace of $V$. Suppose $g_1, \ldots, g_r \in V^*$ forms a basis of the annihilator $W^0$. Write $N_i = \text{Null}(g_i)$. Prove that
   
   $$W = \cap_{i=1}^r N_i.$$  

**Solution:** Write $N = \cap_{i=1}^r N_i$. Suppose $w \in W$. Since $g_i \in W^0$ we have $g_i(w) = 0$. Therefore $w \in N_i$ for $i = 1, \ldots, r$. Hence

$$W \subseteq N.$$  

Now suppose $W \neq N$. Then there is

$$e \in N \quad \text{such that } \quad e \notin W.$$  

We will construct a functional $f \in W^0$ such that $f(e) \neq 0$. Suppose $e_1, \ldots, e_k$ is a basis of $W$. Write $e_{k+1} = e$. Note that $e_1, \ldots, e_k, e_{k+1}$
are linearly independent. Extend this to a basis 
\(e_1, \ldots, e_k, e_{k+1}, e_{k+2}, \ldots, e_n\) of \(V\). Now define \(f : V \rightarrow \mathbb{F}\) by \(f(e_{k+1}) = 1\) and \(f(e_i) = 0\) for \(i = 1, \ldots, n; i \neq k + 1\). (That means \(f = e_{k+1}^*\).)

Then \(f \in W^0\) and \(f(e) \neq 0\). By hypothesis, 
\(f = c_1g_1 + \cdots + c_r g_r\). Since \(e \in N\), we have 
\(g_i(e) = 0\) for \(i = 1, \ldots, r\). Therefore \(f(e) = 0\), 
which is contradiction.

(Sure, I like my proof. But I do not mean that this is the only way or the best way.)

4. Suppose \(V, W\) be two finite dimensional vector spaces over \(\mathbb{F}\).

Let \(T : V \rightarrow W\) be a linear transformation and \(T^t : W^* \rightarrow V^*\) be the transpose. Prove that \(\text{rank}(T) = \text{rank}(T^*)\).

Also prove that for a \(m \times n\) matrix \(A\) with entries in \(\mathbb{F}\), we have \(\text{row} - \text{rank}(A) = \text{column} - \text{rank}(A)\).

5. Suppose \(V\) is vector space of finite dimension, \(\dim V = n\), over \(\mathbb{F}\). Define the map 
\(\varphi : L(V,V) \rightarrow L(V^*,V^*)\)
by \(\varphi(T) = T^t\). Prove that \(\varphi\) is an isomorphism.

**Solution:** Note that
\[\dim(L(V,V)) = \dim(L(V^*,V^*)) = n^2.\]

So, it is enough to prove that \(\varphi\) is one to one. Suppose \(T \in L(V,V)\) and \(\varphi(T) = 0\), we will
prove that \( T = 0 \). Now, we have \( T^t = 0 \). Suppose \( e_1, \ldots, e_n \) is a basis of \( V \) and \( A \) is the matrix of \( T \). Then \( A^t \) is the matrix of \( T^t \) with respect to the dual basis.

Since \( T^t = 0 \), we have \( A^t = 0 \). This implies \( A = 0 \) and hence \( T = 0 \). Hence \( \varphi \) is one to one and therefore an isomorphism.