Math 121 - Answers to Sample Midterm Exam - F07

Part A.

1. (B)	**2.** (C)	**3.** (E)	**4.** (D)	**5.** (B)
6. (E)	**7.** (A)	**8.** (A)	**9.** (B)	**10.** (D)

Part B - Fill in the blanks

11. (A) Because the function is continuous at $x = 1$,

$$\lim_{x\to 1} \frac{2e^x + 1000(\ln x)}{x^2+1} = \frac{2e^1 + 1000(\ln 1)}{1^2+1} = \frac{2e}{2} = e.$$

$\boxed{e}$

(B) Based on the definition of $f(x)$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{0.0001}{x}$.

When $x \to 0^+$, $\frac{1}{x} \to +\infty \Longrightarrow \frac{0.0001}{x} \to \infty$. Hence $\lim_{x\to 0^+} f(x) = \infty$.

(It is considered to be correct if the graph of $f(x)$ is given that shows the answer.)

$\boxed{\infty}$

(C) $\lim_{x\to\infty} \frac{3x^3 - \pi x}{x^2 - \pi x^3} = \lim_{x\to\infty} \frac{3x^3}{-\pi x^3} = \lim_{x\to\infty}\left(-\frac{3}{\pi}\right) = -\frac{3}{\pi}$,

or $\lim_{x\to\infty} \frac{3x^3 - \pi x}{x^2 - \pi x^3} = \lim_{x\to\infty} \frac{3 - \frac{\pi}{x^2}}{\frac{1}{x} - \pi} = \frac{3-0}{0-\pi} = -\frac{3}{\pi}$.

$\boxed{-\frac{3}{\pi}}$

(D) $\lim_{x\to 0} \frac{\sin(\sqrt{2}\,x)}{x} = \lim_{x\to 0} \frac{\sqrt{2}x}{x} = \lim_{x\to 0} \sqrt{2} = \sqrt{2}$,

or $\lim_{x\to 0} \frac{\sin(\sqrt{2}\,x)}{x} = \lim_{x\to 0}\left(\sqrt{2}\,\frac{\sin(\sqrt{2}\,x)}{\sqrt{2}\,x}\right) = \sqrt{2}\lim_{x\to 0}\frac{\sin(\sqrt{2}\,x)}{\sqrt{2}\,x} = \sqrt{2}$.

$\boxed{\sqrt{2}}$

12. (A) The function f is increasing when $f'(x)$ is positive. Since $f'(x) > 0$ on intervals $(0,\,2)$ and $(5,\,9)$, $f(x)$ is increasing on $(0,\,2) \cup (5,\,9)$

$\boxed{(0,\,2) \cup (5,\,9)}$

(B) The function $f(x)$ is concave downward when $f'(x)$ is decreasing (or equivalently, $f''(x) < 0$). Since $f'(x)$ is decreasing on intervals $(0,\,3)$ and $(6,\,7)$, the function $f(x)$ is concave downward on $(0,\,3) \cup (6,\,7)$.

$$\boxed{(0,\,3) \cup (6,\,7)}$$

(C) $f(3)$ is larger than $f(4)$, since $f(x)$ is decreasing on interval $(2,\,5)$ that contains $x = 3, 4$.

$$\boxed{f(3) \text{ is larger}}$$

(D) $f''(4)$ is larger than $f''(3)$. This is because $f''(3)$ and $f''(4)$ are the slopes of the curve of $f'(x)$ at $x = 3, 4$, respectively. Obviously, $f''(4) > 0 = f''(3)$.

$$\boxed{f''(4) \text{is larger}}$$

Part C - Fill in the blanks and justify your answers.

13. In this problem the function f satisfies $\quad f(1) = 2, \quad f(2) = -1; \quad f'(1) = -3, \quad f'(4) = 2.$

(A) Let $G(x) = x^2\,f(x)$. Find $G'(1)$.

$G'(x) = (x^2)'f(x) + x^2 f'(x) = 2xf(x) + x^2 f'(x)$. So $G'(1) = 2(1)f(1) + 1^2 f'(1) = 2(2) + (-3) = 1$.

$$\boxed{G'(1) = 1}$$

(B) Let $H(x) = f(x^2)$. Find the derivative of $H(x)$ at $x = 2$.

$H'(x) = [f'(x^2)](x^2)' = [f'(x^2)](2x) = 2xf'(x^2)$. So $H'(2) = 2(2)f'(2^2) = 4f'(4) = 4(2) = 8$.

$$\boxed{H'(2) = 8}$$

14. (A) Determine the value of constant c in $f(x) = \dfrac{x^2}{x-c}$ such that the graph $f(x)$ has a horizontal tangent line for $x = 4$.

$$f'(x) = \frac{(x^2)'(x-c) - x^2(x-c)'}{(x-c)^2} = \frac{2x(x-c) - x^2}{(x-c)^2} = \frac{x^2 - 2xc}{(x-c)^2}.$$

So $f'(4) = \dfrac{4^2 - 2(4)c}{(4-c)^2} = \dfrac{16 - 8c}{(4-c)^2}$. We need to find the value of c such that $f'(4) = 0$. Then $f'(4) = 0 \Longrightarrow \dfrac{16-8c}{(4-c)^2} = 0 \Rightarrow 16 - 8c = 0 \Longrightarrow c = 2$.

$$\boxed{c = 2}$$

(B) Find an equation of the tangent line to the curve of $xe^y - 2y = 1$ at the point $(1, 0)$.

Using the implicit differentiation,

$$\frac{d}{dx}(xe^y - 2y) = \frac{d}{dx}(1) \Longrightarrow (e^y + xe^y y') - 2y' = 0 \Longrightarrow (2 - xe^y)y' = e^y \Longrightarrow y' = \frac{e^y}{2 - xe^y}.$$

At the point $(1,\,0)$ the corresponding slope is $\dfrac{e^0}{2 - 1e^0} = \dfrac{1}{2-1} = 1$. So the equation of the tangent line is: $y - 0 = 1(x-1) \Rightarrow y = x - 1$.

$$\boxed{y = x - 1}$$

15. (A) Suppose $f(1) = 100$ and $f'(1) = -5$. Use the linearization of $f(x)$ at $x = 1$ to estimate $f(0.96)$.

The linearization is $L(x) = f(1) + f'(1)(x-1) = 100 - 5(x-1) = 105 - 5x$.

So $f(0.96) \approx L(0.96) = 105 - 5(0.96) = 100.2$.

$\boxed{100.2}$

(B) The air is leaking out from a hole of a spherical balloon at a rate of 4π cm^3/s. Find the rate at which the radius of balloon decreases when the radius is 10 cm. (The volume of a balloon with radius r is $V = \frac{4}{3}\pi r^3$.)

At any time, $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$. Since $\frac{dV}{dr} = 4\pi r^2$, when $r = 10$, we have

$\frac{dV}{dr} = 4\pi(10^2) = 400\pi$. Since $\frac{dV}{dt} = -4\pi$, $-4\pi = 400\pi\frac{dr}{dt} \Longrightarrow \frac{dr}{dt} = \frac{-4\pi}{400\pi} = -\frac{1}{100}$.

So the rate at which the radius decreases is $1/100$ cm/s.

$\boxed{\frac{1}{100}\text{cm/s}}$

16. A car is traveling along a straight road. Its position (in feet) after t seconds is given by

$$s(t) = -t^3 + 6t^2.$$

(A) Find the velocity and acceleration at time t.

$v(t) = s'(t) = -3t^2 + 12t, \quad a(t) = v'(t) = -6t + 12.$

$\boxed{v(t) = -3t^2 + 12t \text{ ft/s} \qquad a(t) = -6t + 12\text{ft/s}^2}$

(B) When does the car reach a velocity of 9 ft/s?

$v(t) = 9 \Longrightarrow -3t^2 + 12t = 9 \Longrightarrow t^2 - 4t + 3 = 0 \Longrightarrow (t-1)(t-3) = 0 \Longrightarrow t = 1, 3.$

$\boxed{\text{When} \quad t = 1\,\&\,3\text{s}}$

(C) Is the car speeding up or slowing down after 3 seconds? **Write your answer in the box, and explain in the blank space below.**

$v(3) = -3(3^2) + 12(3) = 9, \qquad a(3) = -6(3) + 12 = -6.$

Since $v(3)$ and $a(3)$ have opposite signs, the car is slowing down after 3 seconds.

$\boxed{\text{slowing down}}$