Modular Catalan Numbers, Generalized Motzkin Numbers, and the Tamari Order

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Joint work with
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• **: arbitrary binary operation on \( \mathbb{C} \)
  
  Notation: \( ab \) means \( a \ast b \).

• Without left-to-right convention, \( abc \) is ambiguous.

• The “product” \( A_n := a_0a_1 \cdots a_n \) has more ambiguity for larger \( n \).

• **Catalan number** \( C_n := \frac{1}{n + 1} \binom{2n}{n} \) measures this ambiguity (for general \( \ast \)) as it enumerates parenthesizations of \( A_n \).
  e.g., \( C_3 = 5 \) counts parenthesizations of 4 factors:
  
  \[
  ((ab)c)d \quad (ab)(cd) \quad (a(bc))d \quad a((bc)d) \quad a(b(cd))
  \]

• Special case: \( \ast \) associative \( \implies \) no ambiguity
  
  \[
  ((ab)c)d = (ab)(cd) = (a(bc))d = a((bc)d) = a(b(cd))
  \]
Use left-to-right convention for ambiguous products.

**Def** $*$ is *k*-associative if for every $A_{k+1}$,

$$(a_0 a_1 \cdots a_k) a_{k+1} = a_0 (a_1 a_2 \cdots a_{k+1})$$

**Examples**

- $a * b := a + b$ is 1-associative because addition is associative.
- $a * b := -a + b$ is 2-associative:

  $$(abc)d = -a + b - c + d = a(bcd)$$

- If $\omega^k = 1$, then $a * b := \omega a + b$ is $k$-associative:

  $$(a_0 a_1 \cdots a_k) a_{k+1} = \omega^{k+1} a_0 + \omega^k a_1 + \cdots + \omega a_k + a_{k+1} = \omega a_0 + \omega^k a_1 + \omega^{k-1} a_2 + \cdots + a_{k+1} = a_0 (a_1 a_2 \cdots a_{k+1})$$

- This talk: $a * b := \omega a + b$ where $\omega$ is a primitive $k$th root of unity.
• Let $x_i$ denote a $\mathbb{C}$-valued variable.

• A parenthesization $P(X_n)$ of $X_n := x_0 x_1 \cdots x_n$ induces a function $\phi_P : \mathbb{C}^{n+1} \to \mathbb{C}$ by $(x_0, \ldots, x_n) \mapsto P(X_n)$.

• Parenthesizations are $k$-equivalent if they induce the same function by the $k$-associativity of $\ast$.

• **Def** The modular Catalan number $C_{k,n}$ enumerates the (distinct) functions of the form $\phi_P : \mathbb{C}^{n+1} \to \mathbb{C}$.

• **Rmk 1** All parenthesizations are 1-equivalent (addition is associative), so $C_{1,n} = 1$.

• **Rmk 2** If $k \geq n$, no parenthesizations are equivalent, so $C_{k,n} = C_n$. 
A few values of $C_{k,n}$

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**Table:** Modular Catalan number $C_{k,n}$ for $n \leq 10$ and $k \leq 8$. 
• $C_n$ counts parenthesizations of $n + 1$ factors ($*$ is applied $n$ times)
• $C_n$ counts (full binary) trees of $n + 1$ leaves ($n$ internal nodes)
• The bijection between these enumerated sets is natural.
• Given by replacing $*$ by $\land$, an operation that joins trees to a common ancestor to form a larger tree.
**Warm up:**

- Which of these parenthesizations are 1-equivalent?
- Which of these parenthesizations are 2-equivalent?
- Which of these parenthesizations are 3-equivalent?
• **Def** A *(left)* *k-comb* is a tree associated to $x_0 x_1 \cdots x_k$.

![Example of a 3-comb](image)

e.g., a 3-comb

• **Def** A *(left)* *k-hook, hook*$_k$, is a tree associated to $x_0 (x_1 x_2 \cdots x_k)$.

![Example of a 3-hook](image)

e.g., the 3-hook

• *k-equivalence* of parenthesizations induces an equivalence relation on trees, also called *k-equivalence*.

**Theorem (H.–Huang)**

$C_{k,n}$ is the number of trees with $n + 1$ leaves with no $k$-hook subtrees.
• **Def** The generalized Motzkin number $M_{k,n}$ is the number of binary trees with $n + 1$ leaves avoiding the $k$-comb as a subtree.

• $M_{3,n}$ is known as the Motzkin number (the number of ways to draw non-intersecting chords between $n$ points of a circle).

• Tree rotations useful for studying $C_{k,n}$ and $M_{k,n}$

• A right (or left) rotation of a tree with 3 leaves changes the structure of the trees without disturbing the order of the leaves:

Let $s$ and $t$ be trees with $n + 1$ leaves. We say $s \succ t$ ($s$ covers $t$) if $s$ may be obtained from $t$ by applying a right rotation to a subtree of $t$. 
• **Example** A tree covering relation:
• Relation $\geq$ generates a partial order on trees with $n + 1$ leaves called the Tamari order.
• e.g., $n = 4$
• Two parenthesizations are equivalent if one may be obtained from the other by a combination of interchanging $k$-combs and $k$-hooks, without changing the order of the other subtrees.

• e.g., a chain of 2-equivalences 4-trees

![Diagram of 4-trees showing 2-equivalences]

• **Rmk** The first interchange is given by applying 2 left rotations and the second is given by applying 2 right rotations.

• **Def** The operation of replacing a $k$-hook subtree by a $k$-comb subtree without altering the rest of the tree is called a *left $k$-rotation* (it is a certain combination of $k$ left rotations).

• **Def** The inverse of a left $k$-rotation is called a *right $k$-rotation*. 
• Write $s \geq_k t$ if tree $s$ may be obtained from tree $t$ by a right $k$-rotation
• Induces the $k$-associative order, weakening of the Tamari order.
• e.g., $n = 4$, $k = 1$ (left) and $k = 2$ (right)
Theorem (H.–Huang)

Each connected component of the $k$-associative poset of $n$-trees has a unique minimal element.

- Note $C_{k,n}$ is the number of minimal elements in the $k$-associative order and $M_{k,n}$ is the number of maximal elements.

Theorem (H.–Huang)

For each $k > 0$, the sequences of modular Catalan numbers and of generalized Motzkin numbers are interlaced:

$$C_{k,1} \leq M_{k,1} \leq C_{k,2} \leq M_{k,2} \leq \cdots C_{k,n} \leq M_{k,n} \leq \cdots.$$
Theorem (Rowland)

If \( t \) is a \( k \)-tree and \( T_n \) is the number of \( n \)-trees that avoid subree \( t \), then the sequence \( \{ T_n \} \) has an algebraic generating function.

Rmks

• Rowland’s proof is constructive, so his methods could be used to find the generating function \( C_k(x) \) of the sequence \( \{ C_{k,n} \}_{n=0}^{\infty} \).
• Instead, we exploit the close relationship between \( C_{k,n} \) and \( M_{k,n} \) to find a polynomial relation on \( C_k(x) \).

Theorem (H.–Huang)

The generating function \( C_k(x) \) of the sequence \( \{ C_{k,n} \}_{n=0}^{\infty} \) satisfies the polynomial equation

\[
x(C_k(x) - 1)^k - xC_k(x)^k + C_k(x)^{k-1} - C_k(x)^{k-2} = 0.
\]
Lagrange inversion give the following:

**Theorem (H.–Huang)**

If \( n \geq 1 \) and \( k \geq 1 \) then

\[
C_{k,n} = \sum_{1 \leq \ell \leq n} \frac{\ell}{n} \sum_{m_1 + \ldots + m_k = n} \sum_{m_2 + 2m_3 + \ldots + (k-1)m_k = n - \ell} \binom{n}{m_1, \ldots, m_k}.
\]

**Rmk** One may write this using the monomial symmetric functions \( m_\lambda \).

\[
C_{k,n} = \sum_{\lambda \subseteq (k-1)^n, \; |\lambda| < n} \frac{n - |\lambda|}{n} m_\lambda(1^n)
\]
Natural question

What binary operations lead to sequences of numbers not described here?

What subtree avoidance rules may be described using binary operations?
Thank you!