12/2/2013

- **Subject**: Applications of Integration
  - Physics
  - Probability
- **Sections**: 6.6; 6.7; 6.8
- **Next**: Summary of what do we know well at the end of the semester.
Great Opportunity:

Extra Credit:

- Reflections from the Math 121 Classroom - 3 weeks before the Final Exam: 10 pts

- Reports from 3 talks with 3 guest speakers, 5 pts each, $\frac{1}{2}$ page

- Participating in the evaluation of lecture and labs, 5 pts
Goals:

- of Students
- of Lab Instructors
- of the Coordinator

Let us think of them.

Thank you!!

Dziekuje!!
Probability - Introduction to Math 526 - Probability and Statistics

Probability density function

\[ f: \quad f(x) \geq 0 \text{ for all } x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \, dx = 1. \]

Example:

Let \( f(x) = \begin{cases} kx^2(1-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases} \)

Q: (a) For what value of \( k \) is \( f \) a probability density function?

(b) For that value of \( k \), find \( P(X \geq \frac{1}{2}) \)

(c) Find the expected value of \( X \).
Solution:

\[ 1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} k x^2 (1-x) \, dx \]

\[ = k \int_{0}^{1} (x^2 - x^3) \, dx = k \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 \]

\[ = k \left[ \left( \frac{1}{3} - \frac{1}{4} \right) - 0 \right] = k \cdot \frac{1}{12} \]

\[ \Rightarrow \quad 1 = k \cdot \frac{1}{12} \quad \iff \quad k = 12 \]

\[ \Rightarrow \quad f(x) = \begin{cases} 12x^2(1-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases} \]

is the probability density function.

(b) \quad P(X \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{1} 12(x^2-x^3) \, dx

\[ = \lim_{t \to \infty} \int_{\frac{1}{2}}^{t} 12(x^2-x^3) \, dx = \lim_{t \to \infty} 12 \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_{\frac{1}{2}}^{t} \]

but observe that \( f(x) = 0 \) for \( x > 1 \)
\[ \int_{\frac{-3}{2}}^{1} 12(x^2-x^3) \, dx \]

\[ = \int_{\frac{-3}{2}}^{1} 12(x^2-x^3) \, dx \]

\[ = 12 \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_{\frac{-3}{2}}^{1} = 12 \left[ \frac{1}{12} - \left( \frac{1}{24} - \frac{1}{64} \right) \right] \]

\[ = 12 \left[ \frac{1}{12} - \frac{1}{24} + \frac{1}{64} \right] \]

\[ = 1 - \frac{1}{2} + \frac{3}{16} \]

\[ = 1 - \frac{8}{16} + \frac{3}{16} = \frac{11}{16} \]

**Recall:**

\[ 0 \leq P(A) \leq 1 \]

with \( P(\emptyset) = 0 \)

and \( P(\Omega) = 1 \)

\[ P(A) = 1 \Rightarrow A = \Omega \]
(c) Expected Value:

$$EX := \int_{-\infty}^{\infty} x f(x) \, dx$$

$$= \int_{0}^{1} 12x (x^2 - x^3) \, dx$$

$$= \int_{0}^{1} 12 (x^3 - x^4) \, dx$$

$$= 12 \left[ \frac{x^4}{4} - \frac{x^5}{5} \right]_{0}^{1} = 12 \left( \frac{1}{4} - \frac{1}{5} \right)$$

$$= 12 \frac{1}{20} = \frac{6}{10} = 0.6$$

*Standard Normal Distribution*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\sigma = 1 \text{ and } \mu = 0$$
Newton's Second Law of Motion:

\[ F = m \frac{d^2s}{dt^2} \]

- **Position** \( s(t) \)
- **Mass** \( m \) in kg
- **Force** in newtons
- **Work done** in moving the object from \( a \) to \( b \):

\[ W = \int_a^b f(x) \, dx \]

- **Force** acts on the object
Example:
A particle is moved along the x-axis by the force that measures \( \frac{10}{(1+x)^2} \) pounds at a point \( x \) feet from the origin. Find the work done in moving the particle from the origin to a distance of 9 ft.

Solution:
\[
W = \int_0^9 \frac{10}{(1+x)^2} \, dx = 10 \left[ -\frac{1}{1+x} \right]_0^9 = 10 \left( -\frac{1}{10} + 1 \right) = \frac{9}{10}
\]

\[
\int \frac{1}{(1+x)^2} \, dx = \int \frac{1}{u^2} \, du = \int u^{-2} \, du = \frac{u^{-1}}{-1} = -\frac{1}{1+x}
\]

\[
1 + x = u \quad \Rightarrow \quad dx = du
\]
Moments and center of mass

Example:
Masses are located at the points \( P_i \):
\[ m_1 = 6, \quad m_2 = 5, \quad m_3 = 10 \]
\[ P_1(1, 5); \quad P_2(3, -2); \quad P_3(-2, -1) \]

Find the moments \( M_x \) and \( M_y \) and the center of mass of the system:

\[
M_x = \sum m_i y_i \\
= 6 \cdot 5 + 5 \cdot (-2) + 10 \cdot (-1) \\
= 10
\]

\[
M_y = \sum m_i x_i \\
= 6 \cdot 1 + 5 \cdot 3 + 10 \cdot (-2) \\
= 6 + 15 - 20 \\
= 1
\]

The center \((\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right)\) is:

\[
= \left( \frac{1}{21}, \frac{10}{21} \right)
\]
Consumer Surplus for the commodity

\[ \int_0^x [p(x) - P] \, dx \]

The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price \( P \) corresponding to an amount demanded of \( x \).

The figure above shows the interpretation of the consumer surplus as the area under the demand curve and above the line \( p = P \).
Example:
A demand curve is given by
\[ p = \frac{450}{x + 8} \]
Find the consumer surplus when the selling price is $10.

Solution:
The consumer surplus is:
\[ \int_{0}^{10} \left( \frac{450}{x + 8} - p \right) \, dx \]
where \( p = p(x) = p(10) = \frac{450}{10 + 8} = \frac{450}{18} = 25 \)

\[ = \int_{0}^{10} \left( \frac{450}{x + 8} - 25 \right) \, dx = \left[ 450 \ln(x + 8) - 45 \right]_{0}^{10} \]
\[ = 450(\ln 18) - 450 - 450 \ln 8 \]
\[ = 450 \left( \ln 18 - \ln 8 - 1 \right) \]
\[ = 450 \left( \ln 2 + 2 \ln 3 - 3 \ln 2 + 1 \right) \]
Review:  -10-

- Sec. 6.6:
  Examples: 1, 6

- Sec. 6.7: 1

- Sec. 6.8: 1, 3, 4.

Q: Which topic from the course is your favorite one? Why?

Essay Question. Think of it.

Q: Which application of calculus is the most fascinating for you? Think of it.