Choose five problems from the following seven:

1. For the following set of data values: 5, 0, 4, 2, 5, 2 find:
   a) mean, \( \bar{x} = \frac{5+0+4+2+5+2}{6} = \frac{18}{6} = 3 \)
   b) median, \( \hat{x} = \frac{2+4}{2} = 3 \)
   c) range, \( R = 5-0 = 5 \)
   d) sample variance, \( S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2 = \frac{1}{5} \sum (0-3)^2 + (2-3)^2 + (4-3)^2 + (5-3)^2 + (5-3)^2 = \frac{20}{5} = 4 \)
   e) sample standard deviation, \( S = \sqrt{S^2} = \sqrt{4} = 2 \)

2. A fair coin is tossed 3 times and the events A and B are defined as follows:

   \[
   \begin{align*}
   A &= \{\text{at least one head is observed}\} = \{HHH, HHT, HTT, THT, TTH, TT, HHT, HTH, HT, TH, TT\} \\
   B &= \{\text{the number of heads observed is odd}\} = \{HHH, HHT, TTT, THT, THH\}
   \end{align*}
   \]

   a) Are A and B disjoint events (mutually exclusive events)? Justify your answer.
   Since \( A \cap B = \{HHH, HHT, TTT, THT, THH\} \neq \emptyset \) 
   \( A, B \) are not disjoint events.

   b) Are A and B independent events? Justify your answer.
   \( A, B \) are independent iff \( P(A\cap B) = P(A) \cdot P(B) \) We check whether this equality holds: \( \frac{4}{8} = P(A\cap B) \neq P(A) \cdot P(B) = \frac{7}{8} \cdot \frac{1}{2} \Rightarrow A, B \) are not independent

   c) Find \( P(B^c) \)
   \[ P(B^c) = 1 - P(B) = 1 - \frac{4}{8} = \frac{4}{8} = \frac{1}{2} \]

   d) Find \( P(A \cap B) \)
   \[ P(A \cap B) = \frac{1}{2} \]

   e) Find \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)
   \[ = \frac{7}{8} + \frac{1}{2} - \frac{1}{2} = \frac{7}{8} \]
3. Let \[ F(x) = \begin{cases} 0 & \text{for } x < 0 \\ cx^2 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases} \]

be a given function.

\[ F(x) \text{ is a (right) continuous function therefore } \lim_{x \to 1^{-}} F(x) = 1 \]

a) Find \( c \) so that \( F \) will be the probability distribution function. Graph \( F \).

\[ \lim_{x \to 1^{-}} F(x) = 1 \quad \Rightarrow \quad \lim_{x \to 1^{-}} F(x) = c \quad \Rightarrow \quad c = 1 \]

b) Find the probability density function \( f \). Graph \( f \).

There is the following relation between \( F(x) \) and \( f(x) \):
\[ f(x) = \frac{d}{dx} F(x) \text{ therefore } f(x) = \int \begin{cases} 2x & 0 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases} \]

\[ f(x) = \begin{cases} 2x & 0 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases} \]

\[ \int_{0}^{1} 2x \, dx = 
\]

\[ \int_{0}^{1} 2x \, dx = \left[ x^2 \right]_{0}^{1} = 1 
\]

\[ \text{c) Find the expected value of } X \text{ and variance of } X. \]
\[ EX = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{1} x \cdot 2x \, dx = 2\left[\frac{x^3}{3}\right]_{0}^{1} = \frac{2}{3} \]

\[ Var(X) = EX^2 - (EX)^2 \]
\[ EX^2 = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{0}^{1} x^2 \cdot 2x \, dx = 2\left[\frac{x^4}{4}\right]_{0}^{1} = \frac{2}{4} = \frac{1}{2} \quad \Rightarrow \quad Var(X) = EX^2 - (EX)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \]

\[ Var(X) = \frac{1}{4} \]

\[ d) \text{ Find } P(X = 1) \text{ and } P(2 < X \leq 3). \]
\[ P(X = 1) = 0 \quad \text{For continuous probability distributions there is probability zero of the random variable assuming any particular value} \]
\[ P(2 < X \leq 3) = \int_{2}^{3} f(x) \, dx = \int_{2}^{3} 0 \, dx = 0 \]
4. The joint probability function of two discrete random variables X and Y is given by the table

<table>
<thead>
<tr>
<th>X/Y</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{3}{10}$</td>
<td>$\frac{2}{10}$</td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td>X:</td>
<td>$\frac{5}{10}$</td>
<td>$\frac{3}{10}$</td>
<td>$\frac{2}{10}$</td>
</tr>
<tr>
<td>Y:</td>
<td>$\frac{6}{10}$</td>
<td>$\frac{4}{10}$</td>
<td></td>
</tr>
</tbody>
</table>

a) i) Find the marginal probability functions of X and Y.

- X:
<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{5}{10}$</td>
<td>$\frac{3}{10}$</td>
<td>$\frac{2}{10}$</td>
</tr>
</tbody>
</table>

- Y:
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{6}{10}$</td>
<td>$\frac{4}{10}$</td>
<td></td>
</tr>
</tbody>
</table>

ii) Are X and Y independent random variables? Justify your answer.

Let's check whether $f(x, y) = f(x) \cdot f(y)$ for all $(x, y)$.

- $\frac{3}{10} = \frac{5}{10} \cdot \frac{6}{10}$ yes
- $\frac{2}{10} \neq \frac{3}{10} \cdot \frac{6}{10}$ does not hold therefore X and Y are not independent.

b) Find EX and EY

- $EX = \sum x_i f(x_i) = 1 \cdot \frac{5}{10} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{2}{10} = \frac{17}{10}$ (17/10)
- $EY = 0 \cdot \frac{6}{10} + 1 \cdot \frac{4}{10} = \frac{4}{10}$ (4/10)

c) i) Find $E(2X) = 2EX = 2 \cdot \frac{17}{10} = \frac{34}{10}$ (34/10)

ii) Find VarX and VarY

- $VarX = EX^2 - (EX)^2 = \left[1^2, \frac{5}{10} + 2^2, \frac{3}{10} + 3^2, \frac{2}{10}\right] - \frac{34}{100} = \frac{35 \cdot 10 - 289}{100} = \frac{61}{100}$
- $VarY = EY^2 - (EY)^2 = \left[0^2, \frac{6}{10} + 1^2, \frac{4}{10}\right] - \frac{16}{100} = \frac{4 \cdot 10 - 16}{100} = \frac{6}{100} = \frac{6}{100}$

iii) Find $Var(2X) = 4 \cdot VarX = 4 \cdot \frac{61}{100} = \frac{61}{25}$

d) Find Cov(X, Y), the covariance of X and Y.

- $Cov(X, Y) = E(XY) - EX \cdot EY = (0, \frac{6}{10} + 1, \frac{2}{10} + 2, \frac{1}{10} + 3, \frac{1}{10}) - \frac{17}{10} \cdot \frac{4}{10}$
- $= \frac{7}{10} - \frac{68}{100} = \frac{70}{100} - \frac{68}{100} = \frac{2}{100} = 0.02$
5. Let \( f(x, y) \) be the joint probability density function of \( X \) and \( Y \) given by

\[
f(x, y) = \begin{cases} \frac{c}{xy} & \text{for } 0 < x < 1 \text{ and } 1 < y < 2 \\ 0 & \text{otherwise} \end{cases}
\]

a) Find the constant \( c. \)

\[
1 = \int_0^1 \int_1^2 \frac{c}{xy} \, dx \, dy = c \int_0^1 \left[ \int_1^2 \frac{1}{xy} \, dy \right] \, dx = c \int_0^1 \left[ \frac{1}{x} \right]_1^2 \, dx = c \int_0^1 \frac{1}{x} \, dx = c \left[ \ln x \right]_0^1 = c \ln 1 - c \ln 0 = \frac{3c}{4}
\]

\[
1 = \frac{3c}{4} \Rightarrow c = \frac{4}{3}
\]

b) Find the marginal probability density functions of \( X \) and \( Y. \)

\[
f_X(x) = \int_1^2 \frac{4}{3xy} \, dy = \frac{4}{3} \int_1^2 \frac{1}{x} \, dx = \frac{4}{3} \left[ \ln x \right]_1^2 = \frac{4}{3} \ln 2 - \frac{4}{3} \ln 1 = \frac{4}{3} \ln 2 \Rightarrow f_X(x) = \begin{cases} \frac{4}{3} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
f_Y(y) = \int_0^1 \frac{4}{3xy} \, dx = \frac{4}{3} \int_0^1 \frac{1}{y} \, dx = \frac{4}{3} \left[ \ln x \right]_0^1 = \frac{4}{3} \ln 1 - \frac{4}{3} \ln 0 = \frac{4}{3} \ln 1 = \frac{4}{3} \Rightarrow f_Y(y) = \begin{cases} \frac{4}{3} & \text{for } 1 < y < 2 \\ 0 & \text{otherwise} \end{cases}
\]

c) Find \( EX \)

\[
EX = \int_0^1 \int_1^2 \frac{4}{3xy} \, dx \, dy = \frac{4}{3} \int_0^1 \int_1^2 \frac{1}{x} \, dx \, dy = \frac{4}{3} \left[ \ln x \right]_1^2 \left[ \ln y \right]_1^2 = \frac{4}{3} \left( \ln 2 \right)^2
\]

\[
EX = \frac{4}{3} \left( \ln 2 \right)^2
\]

d) Find \( E(2X+1) \) and \( E(2X-1)^2 \)

\[
E(2X+1) = 2EX + 1 = 2 \cdot \frac{4}{3} \left( \ln 2 \right)^2 + 1 = \frac{8}{3} \left( \ln 2 \right)^2 + 1 = \frac{8}{3} \ln^2 2 + 1
\]

\[
E(2X-1)^2 = E(4X^2-4X+1) = 4E(X^2) - 4EX + 1 = 4 \int_0^1 \int_1^2 \frac{1}{3x^2} \, dx \, dy - 4 \int_0^1 \int_1^2 \frac{1}{x} \, dx \, dy + 1 = 4 \left[ \frac{1}{3x} \right]_1^2 \left[ \ln x \right]_1^2 = 4 \left( \frac{1}{3} \ln 2 \right)^2 - 4 \left( \frac{1}{3} \ln 1 \right) + 1 = \frac{8}{3} \left( \ln 2 \right)^2 - \frac{8}{3} + 1
\]

e) Are \( X \) and \( Y \) independent random variables?

Yes, because \( f(x, y) = \frac{4}{3}xy = f_X(x) \cdot f_Y(y) = 2x \cdot \frac{4}{3}y = \frac{4}{3}xy \)

6. Let \( A \) and \( B \) be two events with \( P(A) = 0.20, P(B) = 0.60 \) and \( P(A \cap B) = 0.12. \) Find:

a) \( P(B') = 1 - P(B) = 1 - 0.60 = 0.40 \)

b) \( P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.20 + 0.60 - 0.12 = 0.68 \)

c) \( P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{0.12}{0.60} = \frac{12}{60} = \frac{1}{5} \)

d) Are \( A \) and \( B \) independent events? Justify your answer.

Let us check whether \( P(A \cap B) = P(A) \cdot P(B) \)

\[
0.12 = P(A \cap B)^2 = P(A) \cdot P(B) = 0.20 \cdot 0.60 = 0.12
\]

The equality holds, therefore \( A \) and \( B \) are independent.
7. The discrete random variable has the probability distribution given in the table:

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = P(X = x) )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{2}{3} )</td>
</tr>
</tbody>
</table>

a) Find the probability distribution function \( F(x) \).

\[
F(x) = \begin{cases} 
0 & x < 0 \\
\frac{1}{6} & 0 \leq x < 1 \\
\frac{1}{6} + \frac{1}{6} & 1 \leq x < 2 \\
1 & x \geq 2 
\end{cases}
\]

b) Find \( EX \)

\[
EX = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{3} = \frac{1}{6} + \frac{4}{3} = \frac{1}{6} + \frac{8}{6} = \frac{9}{6} = \frac{3}{2}
\]

c) Find \( \text{Var}(X) \)

\[
\text{Var}(X) = EX^2 - (EX)^2 = \left(0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{2}{3}\right) - \left(\frac{3}{2}\right)^2
\]

\[
= \left(\frac{1}{6} + \frac{8}{3}\right) - \frac{9}{4} = \left(\frac{1}{6} + \frac{16}{6}\right) - \frac{9}{4} = \frac{17}{6} - \frac{9}{4} = \frac{68 - 54}{24} = \frac{14}{24} = \frac{7}{12}
\]

d) Find \( P(X \geq 1) \)

\[
P(X \geq 1) = P(X = 1) + P(X = 2)
\]

\[
= \frac{1}{6} + \frac{2}{3} = \frac{1}{6} + \frac{4}{6} = \frac{5}{6}
\]

e) Let \( Y = (-2X + 1) \). Find \( EY \) and \( \text{Var}(Y) \).

\[
EY = E(-2X + 1) = (-2)EX + 1 = (-2) \cdot \frac{3}{2} + 1 = -3 + 1 = -2
\]

\[
\text{Var}(Y) = \text{Var}(-2X + 1) = 4 \cdot \text{Var}X = 4 \cdot \frac{7}{12} = \frac{7}{3}
\]
Extra Credit: \[ a) = 6 \text{ pts} \quad ; \quad b) = 4 \text{ pts} \]

a) A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20% and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

\[
P(D | P_1) = 0.01, \quad P(D | P_2) = 0.03, \quad P(D | P_3) = 0.02,
\]

where \( P(D | P_j) \) is the probability of a defective product, given plan \( j \). If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

Observe that \( P(P_1) = 0.30, \quad P(P_2) = 0.20 \) and \( P(P_3) = 0.50 \)

By Bayes' Rule

\[
P(P_i | D) = \frac{P(D | P_i) \cdot P_i}{P(D)}
\]

for \( i = 1, 2, 3 \)

\[
P(D) = 0.01 \cdot 0.30 + 0.03 \cdot 0.20 + 0.50 \cdot 0.02 = 0.003 + 0.006 + 0.01 = 0.021
\]

\[
P(P_1 | D) = \frac{0.003}{0.021} = \frac{3}{19}
\]

\[
P(P_2 | D) = \frac{0.006}{0.021} = \frac{6}{19}
\]

\[
P(P_3 | D) = \frac{0.01}{0.021} = \frac{10}{19}
\]

\( \Rightarrow \) Plan 3 was most likely used and thus responsible

b) If a random variable \( X \) is defined such that \( E[(X - 1)^2] = 10 \) and \( E[(X - 2)^2] = 6 \), find \( \mu \) and \( \sigma^2 \).

\[
E(X^2 - 2X + 1) = 10 \quad \Rightarrow \quad E(X^2) - 2E(X) + 1 = 10 \quad \Rightarrow \quad E(X^2) - 2E(X) = 9
\]

\[
E(X^2 - 4X + 4) = 6 \quad \Rightarrow \quad E(X^2) - 4E(X) + 4 = 6 \quad \Rightarrow \quad E(X^2) - 4E(X) = 2
\]

\( \Rightarrow \) \( 2E(X) = 7 \) \( \Rightarrow \) \( E(X) = \frac{7}{2} \)

\( \mu = E(X) = \frac{7}{2} \)

\[
E(X^2) = 9 + 2E(X) = 9 + 7 = 16
\]

\( \Rightarrow \) \( \text{Var}(X) = E(X^2) - (E(X))^2 = 16 - \frac{49}{4} = \frac{64 - 49}{4} = \frac{15}{4} \)

\( \Rightarrow \) \( \sigma^2 = \text{Var}(X) = \frac{15}{4} \)