4.2.1 Let the observed value of the mean $\bar{X}$ and of the sample variance of a random sample of size 20 from a distribution that is $N(\mu, \sigma^2)$ be 81.2 and 26.5, respectively. Find respectively 90%, 95%, and 99% confidence intervals for $\mu$.

**Solution.** We use algebra derivation given in Example 4.2.1.
For 90%: $1 - \alpha = .9 \implies \alpha = .1$. There are $20 - 1 = 19$ degrees of freedom. Thus the lower limit is $81.2 - (1.729)(\frac{26.5}{\sqrt{20}}) = 79.209$ and the upper limit is $81.2 + (1.729)(\frac{26.5}{\sqrt{20}}) = 83.190$. So (79.21, 83.19) is the interval with 90% confidence.

For 95%: $1 - \alpha = .95 \implies \alpha = .05$. There are $20 - 1 = 19$ degrees of freedom. Thus the lower limit is $81.2 - (2.093)(\frac{26.5}{\sqrt{20}}) = 78.791$ and the upper limit is $81.2 + (2.093)(\frac{26.5}{\sqrt{20}}) = 83.609$. So (78.79, 83.61) is the interval with 95% confidence.

For 99%: $1 - \alpha = .99 \implies \alpha = .01$. There are $20 - 1 = 19$ degrees of freedom. Thus the lower limit is $81.2 - (2.660)(\frac{26.5}{\sqrt{20}}) = 77.907$ and the upper limit is $81.2 + (2.660)(\frac{26.5}{\sqrt{20}}) = 84.493$. So (77.91, 84.49) is the interval with 95% confidence. \(\square\)

4.2.12 Let $Y$ be $b(300, p)$. If the observed value of $Y$ is $y = 75$, find an approximate 90% confidence interval for $p$.

**Solution.** We have $n = 300$ and a proportion $p = 75/300 = .25$. We have $1 - \alpha = .90 \implies \alpha = .1$. Thus $z_{\alpha/2} = 1.645$. Thus the lower limit is $.25 - (1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .209$ and the upper limit is $.25 + (1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .291$. Hence we have a 90% confidence interval for (.21, .29). \(\square\)

4.2.17 It is known that a random variable $X$ has Poisson distribution with parameter $\mu$. A sample of 200 observation from this distribution has a mean equal to 3.4. Construct an approximate 90% confidence interval for $\mu$.

**Solution.** Since $X$ has the Poisson distribution the mean of $X$ is equal to the variance. We have $1 - \alpha = .9 \implies \alpha = .1$. Thus $z_{\alpha/2} = 1.645$. Then the lower limit is $3.4 - (1.645)(\frac{\sqrt{3.4}}{\sqrt{200}}) = 3.186$ and the upper limit is $3.4 + (1.645)(\frac{\sqrt{3.4}}{\sqrt{200}}) = 3.614$. Hence we have a 90% confidence interval for (3.19,3.61). \(\square\)

4.2.18 Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters $\mu$ and $\sigma$ are unknown. A confidence interval for $\sigma^2$ can be found as follows. We know that $(n-1)S^2/\sigma^2$ is a random variable with $\chi^2(n-1)$ distribution. Thus we can find constants $a$ and $b$ so that $P((n-1)S^2/\sigma^2 < b) = 0.975$ and $P(a < (n-1)S^2/\sigma^2 < b) = 0.95$.

(a) Show that this second probability statement can be written as $P((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95$.

**Proof.** We have

$$a < \frac{(n-1)S^2}{\sigma^2} < b \iff \frac{1}{b} < \frac{\sigma^2}{(n-1)S^2} < \frac{1}{a} \iff \frac{(n-1)S^2}{b} < \sigma^2 < \frac{(n-1)S^2}{a}.\)

Thus $P(a < (n-1)S^2/\sigma^2 < b) = 0.95 \iff P((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95$. \(\square\)

(b) If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for $\sigma^2$.

**Solution.** If $n = 9$ and $s^2 = 7.93$, then $(n-1)S^2 = (9-1)(7.93) = 63.44$, $a = 2.180$, and $b = 17.535$. Thus the lower limit is $63.44/17.535 = 3.618$ and the upper limit is $63.44/2.18 = 29.101$. Thus we have 95% confidence for the interval (3.62, 29.10). \(\square\)
(c) If $\mu$ is known, how would you modify the proceeding procedure for finding a confidence interval $\sigma^2$?

Solution. We know that $\frac{1}{np} \sum_{i=1}^{n} (X_i - \mu)^2$ is $\chi^2(n)$. Then we use $\frac{1}{np} \sum_{i=1}^{n} (X_i - \mu)^2$ in place of $(n-1)s^2/\sigma^2$. △

4.2.21 Let two independent random samples, each of size 10, from two normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ yield $\bar{x} = 4.8$, $s_1^2 = 8.64$, $\bar{y} = 5.6$, $s_2^2 = 7.88$. Find a 95% confidence interval for $\mu_1 - \mu_2$.

Solution. We use equation 4.2.13. We have $n_1 = n_1 = 10$. Thus $n = n_1 + n_2 = 20$. We have $1 - \alpha = .95 \implies \alpha = .05$. Then $t_{.025,18} = 2.101$. Thus the lower limit is

$$(4.8 - 5.6) - (2.101)\sqrt{\frac{(9)(8.64) + (9)(7.88)}{18}} \sqrt{\frac{1}{10} + \frac{1}{10}} = -3.5$$

and the upper limit is

$$(4.8 - 5.6) + (2.101)\sqrt{\frac{(9)(8.64) + (9)(7.88)}{18}} \sqrt{\frac{1}{10} + \frac{1}{10}} = 1.9.$$

Thus we have a 95% confidence interval for $(-3.5, 1.9)$. △

4.2.22 Let two independent random variables, $Y_1$ and $Y_2$, with binomial distributions that have parameters $n_1 = n_2 = 100$, $p_1$, and $p_2$, respectively, be observed to be equal to $y_1 = 50$ and $y_2 = 40$. Determine an approximate 90% confidence interval for $p_1 - p_2$.

Solution. We use equation 4.2.14. We have $p_1 = 50/100 = .5$ and $p_2 = 40/100 = .4$. We find $z_{\alpha/2}$ for $\alpha = .1$. Then $z_{.1/2} = 1.645$. Thus the lower limit is

$$(.5 - .4) - (1.645)\sqrt{\frac{(.5)(.5)}{100} + \frac{(.4)(.6)}{100}} = -.015$$

and the upper limit is

$$(.5 - .4) + (1.645)\sqrt{\frac{(.5)(.5)}{100} + \frac{(.4)(.6)}{100}} = .215.$$

Thus we have a 90% confidence interval for $(-.02, .22)$. △

4.2.27 Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, where the four parameters are unknown. To construct a confidence interval for the ratio, $\sigma_1^2/\sigma_2^2$, of the variances, form the quotient of the two independent $\chi^2$ variables, each divided by its degrees of freedom, namely,

$$F = \frac{(m-1)s_2^2}{(n-1)s_1^2}/(m-1),$$

where $S_1^2$ and $S_2^2$ are the respective sample variances.

(a) What kind of distribution does $F$ have?

Solution. $U = (m-1)s_2^2$ has $\chi^2(m-1)$ distribution and $V = (n-1)s_1^2$ has $\chi^2(n-1)$ distribution. Thus be equation 3.6.7 in the text

$$F = \frac{U/r_1}{V/r_2},$$

where $r_1 = m - 1$ and $r_2 = n - 1$, has $F$-distribution. △

(b) From the appropriate table, $a$ and $b$ can be found so that $P(F < b) = 0.975$ and $P(a < F < b) = 0.95$.

Solution. For $P(F < b) = 0.975$, $b = F_{.025}(m - 1, n - 1)$. For $P(a < F < b) = 0.95$, $b = F_{.05}(m - 1, n - 1)$ and $a = 1/b$. △
(c) Rewrite the second probability statement as

$$P\left[ a \frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b \frac{S_2^2}{S_1^2} \right] = 0.95.$$ 

The observed values, $s_1^2$ and $s_2^2$, can be inserted in these inequalities to provide a 95% confidence interval for $\sigma_1^2/\sigma_2^2$.

Solution. We have

$$F = \frac{(m-1)S_2^2/(m-1)}{(n-1)S_1^2/(n-1)} = S_2^2\frac{\sigma_1^2}{\sigma_2^2}.$$ 

Thus

$$a < \frac{S_2^2\sigma_1^2}{S_1^2\sigma_2^2} < b \iff a \frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b \frac{S_2^2}{S_1^2}.$$ 

Hence

$$P\left[ a \frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b \frac{S_2^2}{S_1^2} \right] = 0.95.$$ 

\[\triangle\]