4.5.8 Let us say the life of a tire in miles, say $X$, is normally distributed with mean $\theta$ and standard deviation 5000. Past experience indicates that $\theta = 30000$. The manufacturer claims that the tires made by a new process have mean $\theta > 30000$. It is possible that $\theta = 35000$. Check his claim by testing $H_0 : \theta = 30000$ against $H_1 : \theta > 30000$. We shall observe $n$ independent values of $X$, say $x_1, \ldots, x_n$, and we shall reject $H_0$ (thus accept $H_1$) if and only if $x \geq c$. Determine $n$ and $c$ so that the power function $\gamma(\theta)$ of the test has the values $\gamma(30000) = 0.01$ and $\gamma(35000) = 0.98$.

**Solution.** To determine $n$ and $c$ we use given conditions for the power function and critical values of normal distribution:

$$
\frac{c - 30000}{5000/\sqrt{n}} = z_{0.01} = 2.326 \quad \text{and} \quad \frac{c - 35000}{5000/\sqrt{n}} = -z_{0.02} = -2.054.
$$

Solving the system we obtain $c = 32655.3$ and $n = 19.184 \approx 19$.

Then, we reject $H_0$ (thus accept $H_1$) if and only if $x \geq 32655.3$.

4.5.9 Let $X$ have a Poisson distribution with mean $\theta$. Consider the simple hypothesis $H_0 : \theta = 1/2$ and the alternative composite hypothesis $H_1 : \theta < 1/2$. Thus $\Omega = \{\theta : 0 < \theta \leq 1/2\}$. Let $X_1, \ldots, X_{12}$ denote a random sample of size 12 from this distribution. We reject $H_0$ if and only if the observed value of $Y = X_1 + \ldots + X_{12} \leq 2$. If $\gamma(\theta)$ is a power function of the test, find the powers $\gamma(1/2)$, $\gamma(1/3)$, $\gamma(1/4)$, $\gamma(1/6)$, and $\gamma(1/12)$. Sketch the graph of $\gamma(\theta)$. What is the significance level of the test?

**Solution.** If random variables $X_1, \ldots, X_{12} \sim \text{Poisson}(\theta)$, then $Y = X_1 + \ldots + X_{12} \sim \text{Poisson}(12 \cdot \theta)$.

Hence, for $\theta = 1/2$ $Y \sim \text{Poisson}(6)$ and

$$
\gamma(1/2) = P(Y \leq 2) = \sum_{k=0}^{2} \frac{6^k}{k!} e^{-6} = e^{-6} \left( 1 + 6 + \frac{6^2}{2} \right) = 0.0619.
$$

For $\theta = 1/3$ $Y \sim \text{Poisson}(4)$ and

$$
\gamma(1/3) = P(Y \leq 2) = \sum_{k=0}^{2} \frac{4^k}{k!} e^{-4} = e^{-4} \left( 1 + 4 + \frac{4^2}{2} \right) = 0.238.
$$

For $\theta = 1/4$ $Y \sim \text{Poisson}(3)$ and

$$
\gamma(1/4) = P(Y \leq 2) = \sum_{k=0}^{2} \frac{3^k}{k!} e^{-3} = e^{-3} \left( 1 + 3 + \frac{3^2}{2} \right) = 0.423.
$$
For $\theta = 1/6$ $Y \sim \text{Poisson}(2)$ and

$$\gamma(1/6) = P(Y \leq 2) = \sum_{k=0}^{2} \frac{2^k}{k!} e^{-2} e^{-2} \left(1 + 2 + \frac{2^2}{2}\right) = 0.677.$$ 

For $\theta = 1/12$ $Y \sim \text{Poisson}(1)$ and

$$\gamma(1/12) = P(Y \leq 2) = \sum_{k=0}^{2} \frac{1^k}{k!} e^{-1} e^{-1} \left(1 + 1 + \frac{1}{2}\right) = 0.919.$$ 

Significance level of the test is $\gamma(1/2) = 0.0619 = 6.2\%$.

---

4.5.10 Let $Y$ have a binomial distribution with parameters $n$ and $p$. We reject $H_0 : p = 1/2$ and accept $H_1 : p > 1/2$ if $Y \geq c$. Find $n$ and $c$ to give a power function $\gamma(p)$ which is such that $\gamma(1/2) = 0.1$ and $\gamma(2/3) = 0.95$, approximately.

**Solution.** To determine $n$ and $c$ we use given conditions for the power function and critical values of normal distribution:

$$\frac{c - np}{\sqrt{np(1-p)}} = \frac{c - n/2}{\sqrt{n/2}} = z_{0.1} = 1.282$$

and

$$\frac{c - 2n/3}{\sqrt{2n/3}} = -z_{0.05} = -1.645.$$ 

Solving the system we obtain $c = 41.562$ and $n = 72.225 \approx 72$. Then power function of the test is

$$\gamma(p) = P(Y \geq c) = P \left( \frac{Y - np}{\sqrt{np(1-p)}} \geq \frac{c - np}{\sqrt{np(1-p)}} \right) =$$

$$= P \left( \frac{Y - 72p}{\sqrt{72p(1-p)}} \geq \frac{41.562 - 72p}{\sqrt{72p(1-p)}} \right), \quad 1/2 \leq p < 1.$$ 

2
4.5.13 Let $p$ denote the probability that, for a particular tennis player, the first serve is good. Since $p = 0.4$, this player decided to take lessons in order to increase $p$. When the lessons are completed, the hypothesis $H_0 : p = 0.4$ will be tested against $H_1 : p > 0.4$ based on $n = 25$ trials. Let $y$ equal the number of first serves that are good, and let the critical region be defined by $C = \{y : y \geq 13\}$.

(a) Determine $\alpha = P(Y \geq 13; p = 0.4)$.

Solution.

$$\alpha = P(Y \geq 13; p = 0.4) = \sum_{k=13}^{n} \binom{n}{k} p^k (1 - p)^{n-k} = \sum_{k=13}^{25} \binom{25}{k} (0.4)^k (0.6)^{25-k} = 0.1537.$$ 

(b) Find $\beta = P(Y < 13)$ when $p = 0.6$; that is, $\beta = P(Y \leq 12; p = 0.6)$ so that $1 - \beta$ is the power at $p = 0.6$.

$$\beta = P(Y \leq 12; p = 0.6) = \sum_{k=1}^{12} \binom{n}{k} p^k (1 - p)^{n-k} = \sum_{k=1}^{12} \binom{25}{k} (0.6)^k (0.4)^{25-k} = 0.1537.$$ 

4.6.4 Consider the one-sided $t$-test for $H_0 : \mu = \mu_0$ versus $H_{A1} : \mu > \mu_0$ constructed in Ex. 5.5.4 and the two-sided $t$-test for $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$ given in (5.6.9). Assume that both tests are of size $\alpha$. Show that for $\mu > \mu_0$, the power function of the one-sided test is larger than the power function of the two-sided test.

Solution. For both tests $X_n \sim N(\mu, \sigma^2)$. The first test’s hypotheses are $H_0 : \mu = \mu_0$ and $H_1 : \mu > \mu_0$, and we reject $H_0$ if $\frac{\mu - \mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha,n-1}$. The second test’s hypotheses are $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$, and we reject $H_0$ if $\left|\frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right| \geq t_{\alpha/2,n-1}$. Then, since $t_{\alpha,n-1} < t_{\alpha/2,n-1}$, for $\mu > \mu_0$

$$\gamma_2(\mu) = P \left( \left|\frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right| \geq t_{\alpha/2,n-1} \right) = P \left( \frac{\mu - \mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha/2,n-1} \right) \leq P \left( \frac{\mu - \mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha,n-1} \right) = \gamma_1(\mu).$$

3
4.6.5 Assume that the weight of cereal in a "10-ounce box" is $N(\mu, \sigma^2)$. To test $H_0: \mu = 10.1$ against $H_1: \mu > 10.1$, we take a random sample of size $n = 16$ and observe that $\bar{x} = 10.4$ and $s = 0.4$.

(a) Do we accept or reject $H_0$ at the 5% significance level?

**Solution.**

$$t_{obs} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{10.4 - 10.1}{0.4/4} = 3.$$ 

Then, since $t_{obs} = 3 > t_{0.05,15} = 1.753$, we reject $H_0$ and accept $H_1$ at the 5% significance level.

(b) What is the approximate $p$-value of this test?

To determine $p$-value of the test we have to find $\alpha$ such that $t_{\alpha,15} \approx 3$. Using table with critical values of $t$ distribution, $p$-value $= \alpha = 0.005$.

4.6.7 Among the data collected for the World Health Organization air quality monitoring project is a measure of suspended particles in $\mu g/m^3$. Let $X$ and $Y$ equal the concentration of suspended particles in $\mu g/m^3$ in the city center (commercial district) for Melbourne and Houston, respectively. Using $n = 13$ observations of $X$ and $m = 16$ observations of $Y$, we shall test $H_0: \mu_X = \mu_Y$ against $H_1: \mu_X < \mu_Y$.

(a) Define the test statistic and critical region, assuming that the unknown variances are equal. Let $\alpha = 0.05$.

**Solution.** The test statistic and the critical region will be

$$T = \frac{Y - X}{S_p \sqrt{1/n_1 + 1/n_2}}, \text{ where } S_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}},$$

and

$$C = \{(X_1, \ldots, X_{13}, Y_1, \ldots, Y_{16}) : T \geq t_{0.05,27} = 1.703\}.$$

(b) If $\bar{x} = 72.9$, $s_x = 25.6$, $\bar{y} = 81.7$, and $s_y = 28.3$, calculate the value of the test statistic and state your conclusion.

$$S_p = \sqrt{\frac{12(25.6)^2 + 15(28.3)^2}{27}} = 27.133,$$

and

$$T_{obs} = \frac{81.7 - 72.9}{27.133 \sqrt{1/13 + 1/16}} = 0.8685.$$
Then, since $T_{obs} < t_{0.05,27} = 1.703$, we do not reject $H_0$ at the significance level $\alpha = 0.05$.

4.6.8 Let $p$ equal the proportion of drivers who use a seat belt in a state that does not have a mandatory seat belt law. It was claimed that $p = 0.14$. An advertising campaign was conducted to increase this proportion. Two months after the campaign, $y = 104$ out of a random sample of $n = 590$ drivers were wearing their seat belts. Was the campaign successful?

(a) Define the null and alternative hypotheses.

**Solution.** Hypotheses are

$$H_0 : p = 0.14 \quad \text{against} \quad H_1 : p > 0.14.$$ 

(b) Define a critical region with an $\alpha = 0.01$ significance level.

The critical region is

$$C = \left\{ y : \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \geq z_{0.01} = 2.326 \right\}, \quad \text{where} \ \hat{p} = \frac{y}{n}.$$ 

(c) Determine the approximate $p$-value and state your conclusion.

$$z_{obs} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.1763 - 0.14}{\sqrt{0.14(0.86)/590}} = 2.539.$$ 

Then, using the table with critical values of normal distribution, correspondent $p$-value $= 0.0055$. And, since $0.0055 < \alpha = 0.01$, we have to reject $H_0$ at the significance level $\alpha = 0.01$.

4.6.9 In exercise 5.4.14 we found a confidence interval for the variance $\sigma^2$ using the variance $S^2$ of a random sample of size $n$ arising from $N(\mu, \sigma^2)$, where the mean $\mu$ is unknown. In testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$, use the critical region defined by $(n-1)S^2/\sigma_0^2 \geq c$. That is, reject $H_0$ and accept $H_1$ if $S^2 \geq c\sigma_0^2(n-1)$. If $n = 13$ and the significance level $\alpha = 0.025$, determine $c$.

**Solution.** Since $\frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1)$, given mentioned conditions,

$$c = \chi^2_{0.025}(12) = 4.404.$$
In exercise 5.4.25, in finding a confidence interval for the ratio of the variances of two normal distributions, we used a statistic $\frac{S_1^2}{S_2^2}$ which has an F-distribution when those two variances are equal. If we denote that statistic by $F$, we can test $H_0: \sigma_1^2 = \sigma_2^2$ against $H_1: \sigma_1^2 > \sigma_2^2$ using the critical region $F \geq c$. If $n = 13$, $m = 11$, and $\alpha = 0.05$, find $c$.

**Solution.** Since $\frac{S_1^2}{S_2^2} \sim F_\alpha(n-1, m-1)$, given mentioned conditions,

$$c = F(12, 10; 0.05) = 2.91.$$