All questions taken from the book.

5.1.3 Let $W_n$ denote a random variable with mean $\mu$ and variance $b/n^p$, where $p > 0$, $\mu$, and $b$ are constants (not functions of $n$). Prove that $W_n$ converges in probability to $\mu$.

**Proof.** Let $\varepsilon > 0$. We have

$$
\lim_{n \to \infty} P\left( |W_n - \mu| \geq \varepsilon \right) \leq \lim_{n \to \infty} \frac{E[|W_n - \mu|^2]}{\varepsilon^2} = \lim_{n \to \infty} \frac{\text{Var}(W_n)}{\varepsilon^2} = \lim_{n \to \infty} \frac{b}{\varepsilon^2 n^p} = 0,
$$

using Chebychev’s inequality where $g(x) = x^2$. Hence $W_n \xrightarrow{p} \mu$. □

4.2.12 Let $Y$ be $b(300, p)$. If the observed value of $Y$ is $y = 75$, find an approximate 90% confidence interval for $p$.

**Solution.** We have $n = 300$ and a proportion $p = 75/300 = .25$. We have $1 - \alpha = .90 \implies \alpha = .1$ Thus $z_{\alpha/2} = 1.645$. Thus the lower limit is $.25 - (1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .209$ and the upper limit is $.25 + (1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .291$. Hence we have a 90% confidence interval for (.21, .29). □

4.2.21 Let two independent random samples, each of size 10, from two normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ yield $x = 4.8, s_1^2 = 8.64, \bar{y} = 5.6, s_2^2 = 7.88$. Find a 95% confidence interval for $\mu_1 - \mu_2$.

**Solution.** We use equation 4.2.13. We have $n_1 = n_2 = 10$. Thus $n = n_1 + n_2 = 20$. We have $1 - \alpha = .95 \implies \alpha = .05$. Then $t_{.05, 18} = 2.101$. Thus the lower limit is

$$(4.8 - 5.6) - (2.101)\sqrt{(9)(8.64) + (9)(7.88)} \sqrt{\frac{1}{10} + \frac{1}{10}} = -3.5$$

and the upper limit is

$$(4.8 - 5.6) + (2.101)\sqrt{(9)(8.64) + (9)(7.88)} \sqrt{\frac{1}{10} + \frac{1}{10}} = 1.9.$$ 

Thus we have a 95% confidence interval for $(-3.5, 1.9)$. □

4.6.08 Let $p$ equal the proportion of drivers who use a seat belt in a country that does not a mandatory seat belt law. It was claimed that $p = .14$. An advertising campaign was conducted to increase this proportion. Two months after the campaign, $y = 104$ out of a random sample of $n = 590$ drivers were wearing their seat belts. Was the campaign succesful?

(a) Define the null and alternative hypotheses.

**Solution.** Define $H_0 : p = .14$ and $H_1 : p > .14$. □

(b) Define a critical region with an $\alpha = .01$ significance level.

**Solution.** The critical region is

$$C = \{ z : z \geq 2.326 \},$$

where

$$z = \frac{y - p}{\sqrt{\frac{p(1-p)}{n}}}.$$ 

(c) Determine the approximate $p$-value and state your conclusion.

**Solution.** We have the observed value of

$$z = \frac{104 - .14}{\sqrt{(.14)(.86)}} = 2.539.$$
Thus $H_0$ is rejected since $2.539 > 2.326$. We conclude that the campaign was successful. △

6.1.02 Let $X_1, X_2, \ldots, X_n$ be a random sample from each of the distributions have the following pdfs:
(a) $f(x; \theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad 0 < \theta < \infty$, zero elsewhere.
(b) $f(x; \theta) = e^{-(x-\theta)}, \quad 0 \leq x < \infty, \quad -\infty < \theta < \infty$, zero elsewhere. Note this is a nonregular case. In each case find the mle $\hat{\theta}$ of $\theta$.

**Solution.**

(a) We have

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}.$$ 

Then

$$l(\theta) = \log[L(\theta)] = \log \left( \prod_{i=1}^{n} \theta x_i^{\theta-1} \right) = n \log(\theta) + \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \log(x_i^{\theta-1})$$

Taking the derivative with respect to $\theta$ and setting equal to zero we have

$$l'(\theta) = 0 \iff \frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i) = 0 \iff \hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \log(x_i)}.$$ 

(b) We have

$$L(\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)},$$ 

if $x_i \geq \theta$ for all $i$, otherwise $L(\theta) = 0$. Then

$$l(\theta) = \log[L(\theta)] = \log \left( \prod_{i=1}^{n} e^{-(x_i-\theta)} \right) = \sum_{i=1}^{n} \log(e^{-(x_i-\theta)}) = - \sum_{i=1}^{n} (x_i - \theta).$$

Taking the derivative with respect to $\theta$ we have

$$l'(\theta) = - \sum_{i=1}^{n} (-1) = n.$$ 

Hence $l(\theta)$ is an increasing function. Thus $\hat{\theta} = \min \{X_1, \ldots, X_n\}$. △

6.2.07 Let $X$ have a gamma distribution with $\alpha = 4$ and $\beta = \theta > 0$.

(a) Find the Fisher information $I(\theta)$.

**Solution.**

We have $f(x; \theta) = \frac{1}{\sqrt{2\pi \theta}} \exp \left( -\frac{x^2}{2\theta} \right)$. The Fisher information is

$$I(\theta) = -E \left[ \frac{\partial^2 \log f(x; \theta)}{\partial \theta^2} \right].$$

Then

$$\log f(x; \theta) = - \log (\Gamma(\alpha)) - \alpha \log(\beta) + \log(x^{\alpha-1}) - \frac{x}{\beta}$$

$$\implies \frac{\partial \log f(x; \theta)}{\partial \theta} = - \frac{\alpha}{\beta} + \frac{x}{\beta^2}$$

$$\implies \frac{\partial^2 \log f(x; \theta)}{\partial^2 \theta} = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3}$$

$$\implies -E \left[ \frac{\partial^2 \log f(x; \theta)}{\partial \theta^2} \right] = - \frac{\alpha}{\beta^2} + 2E[X] = - \frac{\alpha}{\beta^2} + 2\frac{\alpha}{\beta^2} = \frac{\alpha}{\beta^2}.$$ 

Thus $I(\theta) = \frac{\alpha}{\beta^2} = \frac{4}{\theta^2}$. △
(b) If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the mle of $\theta$ is an efficient estimator of $\theta$.

Solution. First note that the mle $\hat{\theta}$ of $\theta$ is $\frac{1}{\alpha} \bar{X}$. Then

$$\text{Var}(\hat{\theta}) = \text{Var}\left(\frac{1}{\alpha} \bar{X}\right) = \frac{1}{\alpha^2} \text{Var}(\bar{X}) = \frac{\beta^2}{\alpha} = \frac{\theta^2}{4}.$$ 

Thus the mle of $\theta$ is an efficient estimator of $\theta$ since $\text{Var}(\hat{\theta}) = \frac{1}{I(\theta)}$. \(\triangle\)