SEMILINEAR STOCHASTIC EQUATIONS IN A HILBERT SPACE WITH A FRACTIONAL BROWNIAN MOTION

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ABSTRACT. The solutions of a family of semilinear stochastic equations in a Hilbert space with a fractional Brownian motion are investigated. The nonlinear term in these equations has primarily only a growth condition. An arbitrary member of the family of fractional Brownian motions can be used in these equations. Existence and uniqueness for both weak and mild solutions are obtained for some of these semilinear equations. The weak solutions are obtained by a measure transformation that verifies absolute continuity with respect to the measure for the solution of the associated linear equation. Some examples of stochastic differential and partial differential equations are given that satisfy the assumptions for the solutions of the semilinear equations.

Key Words: Semilinear stochastic equations, fractional Brownian motion, stochastic partial differential equations, absolute continuity of measures.

1. INTRODUCTION

Fractional Brownian motion denotes a family of Gaussian processes with continuous sample paths that are indexed by the Hurst parameter $H \in (0, 1)$ and that have properties that appear empirically in a wide variety of physical phenomena, such as hydrology, economic data, telecommunications, and medicine. Since some physical phenomena are naturally modeled by stochastic partial differential equations and the randomness can be described by a fractional Gaussian noise, it is important to study the problems of the solutions of stochastic differential equations in a Hilbert space with a fractional Brownian motion. A significant family of these stochastic equations are semilinear equations, so it is important to investigate the existence and the uniqueness of the solutions of the equations and the sample path properties of the solutions. If primarily some growth assumptions are made on the nonlinear terms in the semilinear equations then it is natural to investigate weak solutions, especially those that arise by an absolutely continuous transformation of the measure of the solution of the associated linear stochastic equation.

The study of the solutions of stochastic equations in an infinite dimensional space with a (cylindrical) fractional Brownian motion (for example, stochastic partial differential equations) has been relatively limited. For the Hurst parameter $H \in (1/2, 1)$, linear, bilinear, and semilinear equations research supported in part by NSF grants DMS 0204669, DMS 0505706, ANI 0124510, and GACR 201/04/0750.
with an additive fractional Gaussian noise, the formal derivative of a fractional Brownian motion, are considered in [8, 10, 11, 13, 15, 25]. Random dynamical systems described by such stochastic equations and their fixed points are studied in [22]. A pathwise (or nonprobabilistic) approach is used in [21] to study a parabolic equation with a fractional Gaussian noise where the stochastic term is a nonlinear function of the solution. Strong solutions of bilinear evolution equations with a fractional Brownian motion are considered in [11, 12] and the same type of equation is studied in [30] where a fractional Feynman-Kac formula is obtained. A stochastic wave equation with a fractional Gaussian noise is considered in [2] and a stochastic heat equation with a multiparameter fractional Gaussian noise is studied in [16, 18].

In Section II, some results from fractional calculus are given and these results are used to exhibit a kernel function for an integral operator that provides an isometry of the second moment of Wiener-type stochastic integrals with respect to a fractional Brownian motion and Lebesgue space of square integrable functions. Furthermore, some recent results for the solution of a linear stochastic equation in a Hilbert space ([25]) are described. In Section III, semilinear stochastic equations in a Hilbert space are studied. Initially, an absolute continuity of measures result for transforming the solution of a linear stochastic equation is verified that can be viewed as an analogue of the result of Girsanov ([14]) for a transformation of a finite dimensional standard Brownian motion. For a semilinear stochastic equation where the nonlinear term satisfies a linear growth condition and some additional conditions are satisfied it is shown that there is one and only one weak solution. The weak solution is obtained by verifying an absolute continuity of the measure of the solution with respect to the measure of the solution of the associated linear equation. the cases $H \in (0, 1/2)$ and $H \in (1/2, 1)$ are treated separately. Absolute continuity of the above measures is verified when the nonlinearity satisfies a power growth condition and some additional assumptions are made. In Section IV, some examples of stochastic differential and partial differential equations are given that satisfy the assumptions of the theorems.

2. Preliminaries

In this section, a cylindrical fractional Brownian motion in a separable Hilbert space is introduced, a Wiener-type stochastic integral with respect to this process is defined, and some basic properties of this integral are noted. Initially, some facts from the theory of fractional integration (cf., [28]) are described. Let $(V, \| \cdot \|, \langle \cdot , \cdot \rangle)$ be a separable Hilbert space and let $\alpha \in (0, 1)$. If $\varphi \in L^1([0, T], V)$
then the left-sided and the right-sided fractional (Riemann-Liouville) integrals of \( \varphi \) are defined (for almost all \( t \in [0, T] \)) by

\[
(I_{0+}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) \, ds
\]

and

\[
(I_{T-}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \varphi(s) \, ds
\]

respectively, where \( \Gamma(\cdot) \) is the gamma function. The inverse operators of these fractional integrals are called fractional derivatives and can be given by their respective Weyl representations

\[
(D_{0+}^\alpha \psi)(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{\psi(t)}{t^\alpha} + \alpha \int_0^t \frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}} \, ds \right)
\]

and

\[
(D_{T-}^\alpha \psi)(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{\psi(t)}{(T-t)^\alpha} + \alpha \int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}} \, ds \right)
\]

where \( \psi \in I_{0+}^\alpha (L^1([0,T],V)) \) and \( \psi \in I_{T-}^\alpha (L^1([0,T],V)) \) respectively.

Let \( K_H(t,s) \) for \( 0 \leq s \leq t \leq T \) be the real-valued kernel function

\[
K_H(t,s) = c_H (t-s)^{H-\frac{1}{2}} \frac{2H \Gamma(H+\frac{1}{2})}{\Gamma(2-2H)} \int_s^t (u-s)^{H-\frac{3}{2}} \left( 1 - \frac{s}{u} \right)^{\frac{1}{2}-H} \, du
\]

where

\[
c_H = \left[ \frac{2H \Gamma(H+\frac{1}{2}) \Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \right]^\frac{1}{2}
\]

and \( H \in (0,1/2) \). If \( H \in (1/2,1) \), then \( K_H \) has a simpler form as

\[
K_H(t,s) = c_H \frac{s^{1-H}}{\Gamma(H-\frac{1}{2})} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, du
\]

Define the integral operator \( \mathbb{K}_H \) induced from the kernel \( K_H \) by

\[
\mathbb{K}_H \varphi(t) = \int_0^t K_H(t,s) h(s) \, ds
\]
for \( h \in L^2([0, T], V) \). It is well-known ([28]) that

\[
K_H : L^2([0, T], V) \to \mathcal{I}_{0+}^{H+\frac{1}{2}}(L^2([0, T], V))
\]

is a bijection and \( K_H \) can be described as

\[
K_H h(s) = c_H I_{0+}^1 \left( u_{\frac{1}{2} - H} D_{0+}^{\frac{1}{2} - H} \left( u_{\frac{1}{2} - H} h \right) \right)(s)
\]

for \( H \in (0, 1/2] \) and

\[
K_H h(s) = c_H I_{0+}^1 \left( u_{H - \frac{1}{2}} D_{0+}^{\frac{1}{2} - H} \left( u_{\frac{1}{2} - H} h \right) \right)(s)
\]

for \( H \in [1/2, 1) \) where \( c_H \) is given by (2.2) and

\[ u_\alpha(s) = s^\alpha \]

for \( s \geq 0 \) and \( \alpha \in \mathbb{R} \). The inverse operator

\[
K_H^{-1} : \mathcal{I}_{0+}^{H+\frac{1}{2}}(L^2[0, T], V) \to L^2([0, T], V)
\]

is given by

\[
K_H^{-1} \varphi(s) = c_H^{-1} s^{-\frac{1}{2} - H} D_{0+}^{\frac{1}{2} - H} \left( u_{H - \frac{1}{2}} \varphi \right)(s)
\]

for \( H \in (0, 1/2] \) and

\[
K_H^{-1} \varphi(s) = c_H^{-1} s^{H - \frac{1}{2}} D_{0+}^{\frac{1}{2} - H} \left( u_{\frac{1}{2} - H} \varphi \right)(s)
\]

for \( H \in [1/2, 1) \) and \( \varphi \in \mathcal{I}_{0+}^{H+\frac{1}{2}}(L^2([0, T], V)) \). Note that if \( \varphi \in H^1([0, T], V) \), the Sobolev space, then

\[
K_H^{-1} \varphi(s) = c_H^{-1} s^{H - \frac{1}{2}} I_{0+}^{\frac{1}{2} - H} \left( u_{\frac{1}{2} - H} \varphi' \right)(s)
\]

for \( H \in (0, 1/2] \).

Since the operator \( K_H^{-1} \) plays an important role in the sequel, it is desirable to have some information about its domain \( \mathcal{I}_{0+}^{H+\frac{1}{2}}(L^2([0, T], V)) \). It is straightforward that \( \mathcal{I}_{0+}^{H+\frac{1}{2}}(L^2([0, T], V)) \supset C^\beta([0, T], V) \) for \( \beta > \frac{1}{2} - H \) and \( H \in (0, 1/2] \). However, in Section III, a more general result is needed. If \( H \in (1/2, 1) \), then \( \mathcal{I}_{0+}^{H+\frac{1}{2}}(L^2([0, T], V)) \supset L^2([0, T], V) \).
A definition of the stochastic integral of a deterministic $V$-valued function with respect to a scalar fractional Brownian motion $(\beta(t), t \geq 0)$ is given. This definition uses the methods in [1, 6, 27]. An alternative, equivalent method is given in [11].

Let $\mathcal{K}_H^*: \mathcal{E} \to L^2([0, T], V)$ be the linear map given by

$$
\mathcal{K}_H^* \phi(t) = \phi(t) K_H(T, t) + \int_0^T (\phi(s) - \phi(t)) \frac{\partial K_H}{\partial s}(s, t) \, ds
$$

for $\phi \in \mathcal{E}$ and $K_H$ given by (2.1) where $\mathcal{E}$ is the linear space of $V$-valued step functions on $[0, T]$, that is, $\phi \in \mathcal{E}$ if

$$
\phi(t) = \sum_{i=1}^{n-1} x_i \mathbf{1}_{[t_i, t_{i+1})}(t)
$$

where $x_i \in V$ for $i \in \{1, \ldots, n - 1\}$ and $0 = t_1 < t_2 < \cdots < t_n = T$.

Define the stochastic integral as

$$
\int_0^T \phi d\beta := \sum_{i=1}^n x_i (\beta(t_{i+1}) - \beta(t_i)).
$$

It follows directly that

$$
E \left\| \int_0^T \phi d\beta \right\|^2 = \|\mathcal{K}_H^* \phi\|^2_{L^2([0, T], V)}
$$

where $\| \cdot \|_{L^2([0, T], V)}$ is the norm in $L^2([0, T], V)$ induced by the inner product. Let $(\mathcal{H}, \| \cdot \|, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be the Hilbert space obtained by the completion of the pre-Hilbert space $\mathcal{E}$ with the inner product

$$
\langle \phi, \psi \rangle_{\mathcal{H}} := \langle \mathcal{K}_H^* \phi, \mathcal{K}_H^* \psi \rangle_{L^2([0, T], V)}
$$

for $\phi, \psi \in \mathcal{E}$. The stochastic integral is extended to an arbitrary $\phi \in \mathcal{H}$ by the isometry (2.11). Thus $\mathcal{H}$ is a linear space of integrable functions and it is useful to obtain some more specific information about $\mathcal{H}$. If $H \in (1/2, 1)$ then it is easily verified that $\mathcal{H} \supset \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}}$ is the Banach space of Borel measurable functions with the norm $\| \cdot \|_{\tilde{\mathcal{H}}}$ given by

$$
\| \phi \|^2_{\tilde{\mathcal{H}}} := \int_0^T \int_0^T \| \phi(u) \| \| \phi(v) \| \phi_H(u-v) \, du \, dv
$$

where $\phi_H(u) = H(2H-1)|u|^{2H-2}$ and it is elementary to verify that $\tilde{\mathcal{H}} \supset L^p([0, T], V)$ and, in particular, for $p = 2$ (cf., [12]). If $\phi \in \tilde{\mathcal{H}}$ and $H > 1/2$, then

$$
E \left\| \int_0^T \phi d\beta \right\|^2 = \int_0^T \int_0^T \langle \phi(u), \phi(v) \rangle \phi_H(u-v) \, du \, dv.
$$
If $H \in (0, 1/2)$, then the space of integrable functions is smaller than for $H \in (1/2, 1)$. For $H \in (0, 1/2)$ it is known that $\mathcal{H} \supset H^1([0, T], V)$ (cf., [17, Theorem 5.20]) and $\mathcal{H} \supset C^\beta([0, T], V)$ for each $\beta > 1/2 - H$ (a more specific result is given in the next section). If $H \in (0, 1/2)$, then the linear operator $\mathcal{K}_H$ can be described by a fractional derivative

\begin{equation}
\mathcal{K}_H \phi(t) = c_H t^{1-H} D_{-t}^{1-H} \left( u_{H^{-1/2}} \phi(t) \right)
\end{equation}

where $u_{H^{-1/2}}(s) = s^{H-1/2}$ and its domain is $\mathcal{H} = L^2_{1-H} \left( L^2([0, T], V) \right)$ (cf. [1, Proposition 6]).

A standard cylindrical fractional Brownian motion is defined now.

**Definition 2.1.** Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A cylindrical process $\langle B(\cdot), \cdot \rangle: \Omega \times \mathbb{R}_+ \times V \to \mathbb{R}$ on $(\Omega, \mathcal{F}, P)$ is called a standard cylindrical fractional Brownian motion with the Hurst parameter $H \in (0, 1)$ if

1. For each $x \in V \setminus \{0\}$, $\frac{1}{||x||} \langle B(\cdot), x \rangle$ is a standard scalar fractional Brownian motion with the Hurst parameter $H$.

2. For $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$

   \[ \langle B(t), \alpha x + \beta y \rangle = \alpha \langle B(t), x \rangle + \beta \langle B(t), y \rangle \quad \text{a.s. } P. \]

Note that $\langle B(t), x \rangle$ has the interpretation of the evaluation of the functional $B(t)$ at $x$ though the process $B(\cdot)$ does not take values in $V$.

For $H = 1/2$, this definition is the usual one for a standard cylindrical Wiener process in $V$. For a complete orthonormal basis $(e_n, n \in \mathbb{N})$ of $V$, letting $\beta_n(t) = \langle B(t), e_n \rangle$ for $n \in \mathbb{N}$, the sequence of scalar processes $(\beta_n, n \in \mathbb{N})$ is independent and $B$ can be represented by the formal series

\begin{equation}
B(t) = \sum_{n=1}^{\infty} \beta_n(t) e_n
\end{equation}

that does not converge a.s. in $V$.

Naturally associated with a standard cylindrical fractional Brownian motion is a standard cylindrical Wiener process $(W(t), t \geq 0)$ in $V$ such that, formally, $\dot{B}(t) = \mathcal{K}_H \dot{W}(t)$. For $x \in V \setminus \{0\}$, let $\beta_x(t) = \langle B(t), x \rangle$. It is elementary to verify from (2.1) that there is a scalar Wiener process $(w_x(t), t \geq 0)$ such that

\begin{equation}
\beta_x(t) = \int_0^t K_H(t, s) dw_x(s)
\end{equation}
for $t \in \mathbb{R}_+$. Dually, $w_x(t) = \beta_t \left( (\mathcal{H}_H^*)^{-1} 1_{[0,t]} \right)$ where $\mathcal{H}_H^*$ is given by (2.15) and $V = \mathbb{R}$. Thus there is a formal expansion of $W$,

$$W(t) = \sum_{n=1}^{\infty} w_n(t)e_n$$

(2.18)

where $(e_n, n \in \mathbb{N})$ is a complete orthonormal basis for $V$ and $w_n = \langle W, e_n \rangle$ for $n \in \mathbb{N}$.

Now, the stochastic integral $\int_0^T G dB$ is defined for a suitable operator-valued function $G: [0, T] \to \mathcal{L}(V)$ so that the integral is a $V$-valued random variable.

**Definition 2.2.** Let $G: [0, T] \to \mathcal{L}(V)$ be Borel measurable, $(e_n, n \in \mathbb{N})$ be a complete orthonormal basis in $V$, $G(\cdot)e_n \in \mathcal{H}$ for each $n \in \mathbb{N}$, and $B$ be a standard cylindrical fractional Brownian motion for some fixed $H \in (0, 1)$. The stochastic integral $\int_0^T G dB$ is defined as

$$\int_0^T G dB := \sum_{n=1}^{\infty} \int_0^T Ge_n d\beta_n$$

provided the infinite series converges in $L^2(\Omega, V)$.

The following proposition describes some $\mathcal{L}(V)$-valued functions $G$ that can be used as integrands in Definition 2.2.

**Proposition 2.3.** Let $G: [0, T] \to \mathcal{L}(V)$ be Borel measurable and $G(\cdot)x \in \mathcal{H}$ for each $x \in V$. Let $\Gamma_T: V \to L^2([0, T], V)$ be given as

$$\Gamma_T x (t) = (\mathcal{H}_H^* Gx)(t)$$

(2.20)

for $t \in [0, T]$ and $x \in V$. If $\Gamma_T \in \mathcal{L}_2(V, L^2([0, T], V))$, the linear space of Hilbert-Schmidt operators, then the stochastic integral (2.19) is a centered Gaussian $V$-valued random variable with the covariance operator $\tilde{Q}_T$ given by

$$\tilde{Q}_T x = \int_0^T \sum_{n=1}^{\infty} \langle (\Gamma_T e_n)(s), x \rangle (\Gamma_T e_n)(s) \, ds$$

(2.21)

This integral does not depend on the choice of the complete orthonormal basis $(e_n, n \in \mathbb{N})$ in $V$.

**Proof.** Substituting $G$ in the definition of the stochastic integral (2.19), it is clear that the terms of the summation on the right-hand side are $V$-valued Gaussian random variables by the construction of the integral for a scalar fractional Brownian motion and the sequence of random variables
\[ \left( \int_0^T G e_n \, d \beta_n, n \in \mathbb{N} \right) \] is independent. Computing the second moment of the tail of the series in (2.19) yields

\[
E \left\| \sum_{k=m}^{\infty} G e_k \, d \beta_k \right\|^2 = \sum_{k=m}^{\infty} E \left\| \int_0^T G e_k \, d \beta_k \right\|^2 = \sum_{k=m}^{\infty} \int_0^T \left\| (\mathcal{H}_H^* G e_k) (s) \right\|^2 \, ds
\]

\[
= \sum_{k=m}^{\infty} \int_0^T \left\| (\Gamma_T e_k) (s) \right\|^2 \, ds = \sum_{k=m}^{\infty} \left| \Gamma_T e_k \right|^2_{L^2([0,T],V)}.
\]

It is clear that this final series tends to zero as \( m \) tends to infinity. Thus there is convergence in \( L^2(\Omega, V) \) of the partial sums of the infinite series in (2.19).

To verify that (2.19) is a Gaussian random variable and the form of the covariance \( \tilde{Q}_T \), initially note that for any \( \varphi \in \mathcal{H} \) and \( x \in V \), there is the equality

(2.22) \[
\int_0^T \varphi \, d \beta_x = \int_0^T \mathcal{H}_H^* \varphi (s) \, d w_x
\]

where \( w_x \) is the Wiener process given by (2.17). The terms in the infinite series on the right-hand side of (2.19) are \( V \)-valued, independent centered Gaussian random variables with the sequence of covariance operators \( \left( \tilde{Q}_T^{(n)}, n \in \mathbb{N} \right) \)

(2.23) \[
\tilde{Q}_T^{(n)} x = \int_0^T \langle (\mathcal{H}_H^* G e_n) (s), x \rangle (\mathcal{H}_H^* G e_n) (s) \, ds
\]

for each \( n \in \mathbb{N} \) and \( x \in V \). Thus

(2.24) \[
\tilde{Q}_T x = \sum_{n=1}^{\infty} \int_0^T \langle (\mathcal{H}_H^* G e_n) (s), x \rangle (\mathcal{H}_H^* G e_n) (s) \, ds
\]

\[
= \int_0^T \sum_{n=1}^{\infty} \langle (\Gamma_T e_n) (s), x \rangle (\Gamma_T e_n) (s) \, ds .
\]

The summability of the infinite series on the right-hand side follows from the Hilbert-Schmidt property of \( \Gamma_T \). The independence of the stochastic integral from the choice of the complete orthonormal basis follows from (2.22) and the analogous property for stochastic integrals with respect to a standard cylindrical Wiener process. \( \square \)
It follows by (2.24) that for $x, y \in V$,

$$\langle \tilde{Q}_T x, y \rangle = \int_0^T \sum_{n=1}^{\infty} \langle (\Gamma_T e_n)(s), x \rangle \langle (\Gamma_T e_n)(s), y \rangle \, ds$$

$$= \int_0^T \sum_{n=1}^{\infty} \langle e_n, (\Gamma_T^* x)(s) \rangle \langle e_n, (\Gamma_T^* y)(s) \rangle \, ds$$

$$= \int_0^T \langle (\Gamma_T^* x)(s), (\Gamma_T^* y)(s) \rangle \, ds$$

$$= \int_0^T \langle \Gamma_T (\Gamma_T^* x)(s), y \rangle \, ds.$$

If $H \in (1/2, 1)$, then $\tilde{Q}_T$ satisfies

$$\tilde{Q}_T = \int_0^T \int_0^T G(u) G^*(v) \phi_H(u-v) \, du \, dv$$

where $\phi_H(u) = H(2H - 1)|u|^{2H-2}$ and $G$ is assumed to satisfy

$$\int_0^T \int_0^T |G(u)|_{\mathcal{L}_2(V)} |G(v)|_{\mathcal{L}_2(V)} \phi_H(u-v) \, du \, dv < \infty$$

(cf. [13, Proposition 2.2]).

The next proposition shows that some densely defined linear operators commute with the stochastic integration.

**Proposition 2.4.** If $\tilde{A} : \text{Dom}(\tilde{A}) \to V$ is a closed linear operator, $\text{Dom}(\tilde{A}) \subset V$, and $G : [0, T] \to \mathcal{L}(V)$ is Borel measurable such that $G([0, T]) \subset \text{Dom}(\tilde{A})$ and both $G$ and $\tilde{A}G$ satisfy the conditions for $G$ in Proposition 2.3, then

$$\int_0^T G dB \subset \text{Dom}(\tilde{A}) \quad \text{a.s. } P$$

and

$$\tilde{A} \int_0^T G dB = \int_0^T \tilde{A} G dB \quad \text{a.s. } P. \tag{2.25}$$

**Proof.** By the assumptions on $G$ and $\tilde{A}G$, it follows that $Ge_n \in \mathcal{H}$ and $\tilde{A}Ge_n \in \mathcal{H}$ for $n \in \mathbb{N}$ so by a standard argument using a sequence of step functions, the following equality is satisfied:

$$\tilde{A} \int_0^T Ge_n d\beta_n = \int_0^T \tilde{A} Ge_n d\beta_n.$$
Since the sequence of integrals are Gaussian random variables it follows that

\[
\lim_{m \to \infty} \sum_{n=1}^{m} \int_0^T G e_n \, d\beta_n = \int_0^T G dB \quad \text{a.s. } \mathbb{P}
\]

and

\[
\lim_{m \to \infty} \tilde{A} \left( \sum_{n=1}^{m} \int_0^T G e_n \, d\beta_n \right) = \lim_{m \to \infty} \sum_{n=1}^{m} \int_0^T \tilde{A} G e_n \, d\beta_n = \int_0^T \tilde{A} G dB \quad \text{a.s. } \mathbb{P}.
\]

Since \( \tilde{A} \) is a closed linear operator it follows that \( \int_0^T G dB \in \text{Dom}(\tilde{A}) \) a.s. \( \mathbb{P} \) and the equality (2.25) is satisfied. \( \square \)

Some results are reviewed for a linear stochastic differential equation with a cylindrical fractional Brownian motion whose solution is often called a fractional Ornstein-Uhlenbeck process. This process is a mild solution of the linear stochastic equation

\[
dZ(t) = AZ(t) \, dt + \Phi dB(t)
\]

\[Z(0) = x\]

where \( Z(t), x \in V, (B(t), t \geq 0) \) is a standard cylindrical fractional Brownian with \( H \in (0, 1) \), \( \Phi \in \mathcal{L}(V), A : \text{Dom}(A) \to V, \text{Dom}(A) \subset V \), and \( A \) is the infinitesimal generator of a strongly continuous semigroup \( (S(t), t \geq 0) \) on \( V \). A mild solution of (2.27) is

\[
Z(t) = S(t)x + \int_0^t S(t-r) \Phi dB(r)
\]

\[= S(t)x + \hat{Z}(t)\]

where the stochastic integral in (2.28) is given by Definition 2.2.

Typically it is assumed that \( (S(t), t \geq 0) \) is an analytic semigroup. In this case, there is a \( \hat{\beta} \in \mathbb{R} \) such that the operator \( \hat{\beta} I - A \) is uniformly positive on \( V \). For each \( \delta \geq 0 \), \( (V_\delta, \| \cdot \|_\delta) \) is a Hilbert space where \( V_\delta = \text{Dom}\left(\left(\hat{\beta} I - A\right)^\delta\right) \) with the graph norm topology so that

\[
\|x\|_\delta = \left\|\left(\hat{\beta} I - A\right)^\delta x\right\|
\]

For the mild solution of (2.27), the cases \( H \in (0, 1/2) \) and \( H \in (1/2, 1) \) have been treated separately [13, 25] because the conditions for similar results are somewhat different. The case \( H = 1/2 \) (Brownian motion) has been studied extensively (cf., [4]).

For \( H \in (1/2, 1) \), the following sample path property of the solution is described in [13].
Proposition 2.5. If $H \in (1/2, 1)$, $S(t) \Phi \in \mathcal{L}_2(V)$ for each $t > 0$ and

$$
(2.29) \quad \int_0^{T_0} \int_0^{T_0} u^{-\alpha} v^{-\alpha} |S(u)\Phi|_{\mathcal{L}_2(V)} |S(v)\Phi|_{\mathcal{L}_2(V)} \Phi_H(u-v) \, du \, dv < \infty
$$

for some $T_0 > 0$ and $\alpha > 0$ then there is a Hölder continuous $V$-valued version of the process $(\hat{Z}(t), t \geq 0)$ with Hölder exponent $\beta < \alpha$ where $\hat{Z}$ is the stochastic convolution in (2.28). If $(S(t), t \geq 0)$ is an analytic semigroup then there is a version of the process $(\hat{Z}(t), t \in [0, T])$ with $C^\beta([0, T], V_\delta)$ sample paths for each $T > 0$ and $\beta + \delta < \alpha$.

For $H \in (0, 1/2)$, there are the following results for the sample path behavior of the mild solution ([25]):

Proposition 2.6. Let $(S(t), t \geq 0)$ be an analytic semigroup, $H \in (0, 1/2)$ and

$$
(2.30) \quad |S(t)\Phi|_{\mathcal{L}_2(V)} \leq ct^{-\gamma}
$$

for $t \in [0, T]$, some $c > 0$, and $\gamma \in [0, H)$. Let $\alpha \geq 0$ and $\delta \geq 0$ satisfy

$$
(2.31) \quad \alpha + \beta + \gamma < H,
$$

then there is a version of the process $(\hat{Z}(t), t \in [0, T])$ with $C^\alpha([0, T], V_\delta)$ sample paths. If it is assumed instead of (2.30) and (2.31) that $\Phi \in \mathcal{L}_2(V)$ and $\alpha + \delta < H$ then the process $(\hat{Z}(t), t \in [0, T])$ has a $C^\alpha([0, T], V_\delta)$ version. In particular, there is a $C^\alpha([0, T], V)$ version for $0 < \alpha < H$.

Proposition 2.6 is also valid for $H \in (1/2, 1)$ ([25]).

3. Semilinear Stochastic Equations

In this section, both weak and mild solutions are obtained for various semilinear stochastic equations with a fractional Brownian motion. The cases $H \in (0, 1/2)$ and $H \in (1/2, 1)$ are treated separately as in the case of the linear stochastic equations (Proposition 2.5 and Proposition 2.6). The weak solution of a semilinear equation is obtained by an absolutely continuous transformation of the measure for the solution of the associated linear equation. The absolute continuity methods given here are an analogue of the results for the measure of a finite dimensional fractional Brownian motion ([7, 9, 13, 23, 24]) and the results for Wiener measure ([3, 14]). For a fixed $H \in (0, 1)$ and $T > 0$, let $(\mathcal{F}_t, t \in [0, T])$ be the filtration for the standard cylindrical fractional Brownian motion $(B(t), t \in [0, T])$ with the Hurst parameter $H$. The sub-$\sigma$-algebra $\mathcal{F}_t \subset \mathcal{F}$ can be generated
by \( \sigma (\beta_n(s), s \in [0,t], n \in \mathbb{N}) \) where \( (\beta_n, n \in \mathbb{N}) \) is a sequence of independent scalar fractional Brownian motions with the Hurst parameter \( H \) that is given in the definition of a standard cylindrical fractional Brownian motion (Definition 2.1).

The following result describes an absolute continuity for a transformation of a standard cylindrical fractional Brownian motion.

**Theorem 3.1.** Let \( H \in (0,1) \) and \( T > 0 \) be fixed and let \( (u(t), t \in [0,T]) \) be a \( V \)-valued, \((\mathcal{F}_t)\)-adapted process such that

1. \[ \int_0^T \|u(t)\| dt < \infty \quad \text{a.s. } \mathbb{P} \]

and

2. \[ U(\cdot) := \int_0^\cdot u(s) ds \in L_0^{H+\frac{1}{2}} (L^2([0,T],V)) \quad \text{a.s. } \mathbb{P} . \]

Furthermore, it is assumed that

\[ \mathbb{E} \xi(T) = 1 \]

where

\[ \xi(T) = \exp \left[ \int_0^T \langle \mathbb{K}_H^{-1}(U)(t), dW(t) \rangle - \frac{1}{2} \int_0^T \| \mathbb{K}_H^{-1}(U)(t) \|^2 dt \right] \]

where \((W(t), t \in [0,T])\) is a standard cylindrical Wiener process in \( V \) given by (2.18) and \( \mathbb{K}_H^{-1} \) is the inverse of the integral operator \( \mathbb{K}_H \) in (2.4). Then the process \((\tilde{B}(t), t \in [0,T])\) given by

\[ \tilde{B}(t) := B(t) - U(t) \]

is a standard cylindrical fractional Brownian motion in \( V \) with the Hurst parameter \( H \) on the probability space \((\Omega, \mathcal{F}, \tilde{\mathbb{P}})\) where

\[ \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \xi(T) \quad \text{a.s.} \]

**Proof.** Initially, it is noted that for an \((\mathcal{F}_t)\)-adapted process, \((\eta(t), t \in [0,T])\) with \( \eta \in L^2([0,T],V) \) a.s. \( \mathbb{P} \), \( \int_0^T \langle \eta, dW \rangle \) is defined by

\[ \int_0^T \langle \eta, dW \rangle = \sum_{n=1}^\infty \int_0^T \langle \eta, e_n \rangle \, dw_n \]
where the sequences \((\beta_n, n \in \mathbb{N})\) and \((w_n, n \in \mathbb{N})\) are related by (2.17). It is shown that \(K_H^{-1}U\) satisfies the conditions of \(\eta\) so that the stochastic integral in (2.19) is well-defined. Recall that the linear operator \(K_H\) given in (2.4) is a bijection

\[
K_H : L^2([0, T], V) \rightarrow L^{H+\frac{1}{2}}_0 ([0, T], V)
\]

so by assumption 1 in Theorem 3.1, \(K_H^{-1}(U) \in L^2([0, T], V)\) a.s. \(P\). From the definition of \(K_H\), it follows that \((K_H^{-1}(U)(t), t \in [0, T])\) is an \((\mathcal{F}_t)\)-adapted process because \(U\) is \((\mathcal{F}_t)\)-adapted. By the construction of the standard cylindrical Wiener process \(W\), it is a Wiener process with respect to \((\mathcal{F}_t)\) so \(\xi_T\) is a well-defined random variable. By a Girsanov theorem for Wiener processes in infinite dimensions (cf., [4]), the equality (3.2) defines a probability \(\tilde{P}\) on \((\Omega, \mathcal{F})\) such that

\[
\tilde{W}(t) := W(t) - \int_0^t K_H^{-1}(U)(s) ds
\]

is a standard cylindrical Wiener process in \(V\). Let

\[
\tilde{\beta}_n(t) := \langle B(t), e_n \rangle - \langle U(t), e_n \rangle
\]

and

\[
\tilde{w}_n(t) = \langle W(t), e_n \rangle - \left\langle \int_0^t K_H^{-1}(U)(s) ds, e_n \right\rangle.
\]

It follows that

\[
(3.3) \quad \int_0^t K_H(t, s) d\tilde{w}_n(s) = \int_0^t K_H(t, s) dw_n(s) - \int_0^t K_H(t, s) \langle K_H^{-1}(U)(s), e_n \rangle ds
\]

\[
= \beta_n(t) - \left\langle \int_0^t K_H(t, s) \langle K_H^{-1}(U)(s), e_n \rangle ds, e_n \right\rangle
\]

\[
= \beta_n(t) - \langle K_H K_H^{-1}(U)(t), e_n \rangle
\]

\[
= \beta_n(t) - \langle U(t), e_n \rangle = \tilde{\beta}_n(t).
\]

Thus \((\tilde{B}(t), t \in [0, T])\) is a standard cylindrical fractional Brownian motion in \(V\) with the Hurst parameter \(H\) on \((\Omega, \mathcal{F}, \tilde{P})\). □

In this section, the following semilinear stochastic equation is considered:

\[
(3.4) \quad dX(t) = (AX(t) + F(X(t))) dt + \Phi dB(t)
\]
where $t \in \mathbb{R}_+$, $X(t)$, $X_0 \in V$, $(B(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion with the Hurst parameter $H \in (0,1)$, $\Phi \in \mathcal{L}(V)$, $A: \text{Dom}(A) \to V$, $\text{Dom}(A) \subseteq V$, and $A$ is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on $V$. The function $F: V \to V$ is nonlinear and for the applications to stochastic partial differential equations it is more useful to assume that $F$ is only defined on a (dense) subspace of $V$. So, let $(E, \| \cdot \|_E)$ be a separable Banach space that is continuously embedded in $V$ and $F: E \to V$ with $X_0 \in E$. It is assumed that $F: E \to V$ is Borel measurable, $\text{Im}(F) \subset \text{Im}(\Phi)$, for $G := \Phi^{-1}F$, $G: E \to V$, and

$$\tag{3.5} \|G(x)\| \leq \hat{k} \left(1 + \|x\|_E^\rho \right)$$

and

$$\tag{3.6} \|F(x)\|_E \leq \hat{k} \left(1 + \|x\|_E^\rho \right)$$

for each $x \in E$ and some $\rho \geq 1$. Furthermore, it is assumed that there is a constant $\bar{K}$ such that for each pair $(x, y)$ in $\text{Dom}(A)$, there is a $z^* \in \partial \|z\|_E$ such that

$$\tag{3.7} \langle Ax - Ay + F(x) - F(y), z^* \rangle_{E, E^*} \leq \bar{K}\|x - y\|_E$$

where $\partial \|z\|_E$ is the subdifferential of the norm $\|z\|_E$ at the point $z = x - y$ and $\langle \cdot, \cdot \rangle_{E, E^*}$ is the pairing between $E$ and $E^*$. The inequality (3.7) is a one-sided growth condition that ensures the absence of explosions of solutions of (3.4) in a finite time. Some subsequent examples should clarify its interpretation.

The notions of a weak and a mild solution of (3.4) are given now.

**Definition 3.2.** A weak solution of the equation (3.4) is a triple $(X(t), B(t), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), t \geq 0)$ where $(B(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion in $V$ that is defined on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $(X(t), t \geq 0)$ is an $E$-valued process satisfying

$$\tag{3.8} X(t) = S(t)X_0 + \int_0^t S(t-r)F(X(r)) \, dr + \int_0^t S(t-r)\Phi \, dB(r) \ .$$

A mild solution, $(X(t), t \geq 0)$ of the equation (3.4) is an $E$-valued process on a fixed probability space $(\Omega, \mathcal{F}, P)$ with a given standard cylindrical fractional Brownian motion that is the fractional Brownian motion in (3.8), and the process $(X(t), t \geq 0)$ satisfies (3.8).
The equation (3.8) has a unique weak solution if for any two weak solutions \((X(t), B(t), (\Omega, \mathcal{F}, \mathbb{P}), t \geq 0)\) and \((\tilde{X}(t), \tilde{B}(t), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), t \geq 0)\), the processes \((X(t), t \geq 0)\) and \((\tilde{X}(t), t \geq 0)\) have the same probability law.

The equation has a unique mild solution if for any two processes \((X_1(t), t \geq 0)\) and \((X_2(t), t \geq 0)\) that satisfy (3.4) on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the same standard cylindrical fractional Brownian motion, \(\mathbb{P}(X_1(t) = X_2(t), t \geq 0) = 1\).

A primary goal in this section is to verify weak existence and weak uniqueness of a solution of (3.4). Since the cases \(H \in (0, 1/2)\) and \(H \in (1/2, 1)\) require different methods, they are treated separately.

The following three assumptions are made to construct a solution of (3.4):

(H1). The semigroup \((S(t), t \geq 0)\) generated by \(A\) is analytic on \(V\) and for each \(t \geq 0\), \(S(t)|_E \in \mathcal{L}(E)\) and \(\|S(t)\|_{\mathcal{L}(E)}\) is bounded on compact time intervals.

(H2). \(\Phi \in \mathcal{L}(V)\) is injective and for \(T > 0\), the stochastic convolution process
\[
\left( \int_0^T S(t-r)\Phi dB(r), t \in [0, T] \right)
\]
has a version with \(C([0, T], E)\) sample paths.

(H3). The function \(F : E \to V\) in (3.4) is Borel measurable, \(\text{Im}(F) \subset \text{Im}(\Phi)\) and the function \(G = \Phi^{-1}F : E \to V\) satisfies
\[
\|G(x)\| \leq k (1 + \|x\|_E)
\]
for some \(k > 0\) and all \(x \in E\).

The following result verifies a weak solution for \(H \in (0, 1/2)\).

**Theorem 3.3.** If \(H \in (0, 1/2)\) and conditions (H1)-(H3) are satisfied, then the equation (3.4) has a weak solution. If additionally \(F : E \to E\) and
\[
\|F(x)\|_E \leq k_1 (1 + \|x\|_E)
\]
for some \(k_1 > 0\) and all \(x \in E\), then the weak solution is unique.

**Proof.** Initially, existence of a weak solution is verified. By a standard method that has been used for equations of the form (3.4) with a standard cylindrical Brownian motion (cf., [4]), it suffices to
verify that the cylindrical process

\[
\dot{B}(t) = B(t) - \int_0^t G(Z(s)) \, ds
\]

is a standard cylindrical fractional Brownian motion in a suitable probability space where

\[
Z(t) = S(t)X_0 + \tilde{Z}(t)
\]

satisfies the associated linear equation. To use Theorem 3.1 it is necessary to verify that \(G = \Phi^{-1}F\) satisfies the conditions of \(U\) in this theorem, that is,

\[
\int_0^T G(Z(s)) \, ds \in L_{0+}^{H+\frac{1}{2}} \left(L^2([0,T],V)\right) \tag{3.11}
\]

and

\[
\mathbb{E} \exp[\rho(Z)] = 1 \tag{3.12}
\]

where

\[
\rho(Z) = \int_0^T \left\langle K^{-1}_H \left( \int_0^t G(Z) \right)(t), dW(t) \right\rangle - \frac{1}{2} \int_0^T \left\| K^{-1}_H \left( \int_0^t G(Z) \right)(t) \right\|^2 dt, \tag{3.13}
\]

\(K^{-1}_H\) is the inverse of \(K_H\) in (2.4) and \((W(t), t \geq 0)\) is a standard cylindrical Wiener process in \(V\) by (2.18).

From (2.5), it follows that

\[
\left| K^{-1}_H \left( \int_0^t G(Z) \right) \right|_{L^2([0,T],V)}^2 = c_H^{-1} \left| u_{H-\frac{1}{2}} L_0^{1-H} \left( u_{\frac{1}{2}-H} G(Z) \right) \right|_{L^2([0,T],V)}^2
\]

\[
= \hat{c}_H \int_0^T \left( s^{H-\frac{1}{2}} \left\| \int_0^s r^{\frac{1}{2}-H} (s-r)^{-\frac{1}{2}-H} G(\tilde{Z}(r)) \, dr \right\| \right)^2 ds \tag{3.14}
\]

\[
\leq \hat{c}_H k^2 \left( 1 + |\tilde{Z}|_{C([0,T],E)} \right) + \sup_{r \in [0,T]} \|S(t)X_0\|_E \int_0^T s^{2H-1} \left( \int_0^s r^{\frac{1}{2}-H} (s-r)^{-\frac{1}{2}-H} \, dr \right)^2 ds
\]

\[
\leq c_T \left( 1 + |\tilde{Z}|_{C([0,T],E)}^2 \right)
\]
for some $c_T > 0$. This inequality verifies (3.11). The following Novikov condition is used to verify (3.12):

\begin{equation}
E \exp \left[ \hat{k} \int_0^T \left\| K_H^{-1} \left( \int_0^t G(Z) \right)(t) \right\|^2 dt \right] \leq c E \exp \left[ \hat{k} c_T \tilde{Z}^2_{C([0,T],E)} \right]
\end{equation}

for some $c > 0$. Since $c_T \downarrow 0$ as $T \downarrow 0$ and $z$ is a $C([0,T],E)$-valued Gaussian random variable, it follows that

\begin{equation}
E \exp \left[ \hat{k} c_T \tilde{Z}^2_{C([0,T],E)} \right] < \infty
\end{equation}

for $T > 0$ sufficiently small by the Fernique inequality. For arbitrary $T > 0$, a simple iteration verifies the result, that is,

\begin{equation}
E \exp \left[ \hat{k} \int_{T_{m-1}}^{T_m} \left\| K_H^{-1} \left( \int_0^t G(Z) \right)(t) \right\|^2 dt \right] < \infty
\end{equation}

for a sufficiently fine partition $0 = T_0 < T_1 < \cdots < T_n = T$. The $\sigma$-algebra $\sigma(W(s), s \in [0, T])$ is generated by sets $(A_1, A_2, \ldots, A_n)$ where $A_i \in \sigma(W((s), s \in [T_{i-1}, T_i])$.

Now, uniqueness of the weak solution is verified. Uniqueness in law can be proved in a standard way (cf., [19, Proposition 5.3.10]) by removing the term $F$ in (3.4) by absolute continuity of measures which is a suitable inverse of the above construction of a weak solution.

Let $(\tilde{X}(t), t \in [0, T])$ be a solution to the equation

\begin{equation}
\tilde{X}(t) = S(t)\tilde{x}_0 + \int_0^t S(t-r)F(X(r)) dr + \tilde{Z}(t)
\end{equation}

where $\tilde{Z}(t) = \int_0^t S(t-r)\Phi dB(r)$ and $(B(t), t \in [0, T])$ is some standard cylindrical fractional Brownian motion on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

It suffices to assume that $\tilde{\Omega} = C([0, T], V)$ and $\tilde{X}$ is the canonical process and thereby to show that

\begin{equation}
\exp \left[ \hat{\rho}(\tilde{X}) \right] := \exp \left[ - \int_0^T \left\langle K_H^{-1} \left( \int_0^t G(\tilde{X}) \right)(t), dW(t) \right\rangle - \frac{1}{2} \int_0^T \left\| K_H^{-1} \left( \int_0^t G(\tilde{X}) \right)(t) \right\|^2 dt \right]
\end{equation}
is a Radon-Nikodym derivative on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) so \(\tilde{P}\) is the measure for a fractional Ornstein-Uhlenbeck process and uniqueness in law follows. Thus it is necessary to show that

\[ (3.20) \quad \int_0^t G(\tilde{X}(s)) \, ds \in \mathcal{L}^{H+\frac{1}{2}}_0([0,T], V) \]

and

\[ (3.21) \quad \tilde{E} \exp[\tilde{\rho}(\tilde{X})] = 1 \]

where \(\tilde{E}\) is integration with respect to \(\tilde{P}\). The verifications of (3.20) and (3.21) are analogous to the verifications of (3.11) and (3.12) respectively. However, since \(\tilde{X}\) is not a Gaussian process, the Fernique inequality cannot be used directly. Initially, it is verified that there is a \(c > 0\) such that

\[ (3.22) \quad |\tilde{X}|_{C([0,T], E)} \leq c \left( 1 + \|X_0\|_{E} + |\tilde{Z}|_{C([0,T], E)} \right) . \]

Let

\[
    u(t) = \tilde{X}(t) - \tilde{Z}(t) = S(t)X_0 + \int_0^t S(t-r)F(u(r) + \tilde{Z}(r)) \, dr .
\]

Thus

\[ (3.23) \quad \|u(t)\|_{E} \leq c_1 \|X_0\|_{E} + c_2 \int_0^t \left( 1 + \|u(r)\|_{E} + \|\tilde{Z}(r)\|_{E} \right) \, dr \]

for some positive constants \(c_1\) and \(c_2\). By the Gronwall Lemma it follows that

\[ (3.24) \quad \|u(t)\|_{E} \leq c_1 \left( 1 + \|X_0\|_{E} + |\tilde{Z}|_{C([0,T], E)} \right) \]

for \(t \in [0, T]\) so the inequality (3.22) is verified. The exponential that usually occurs in the Gronwall inequality is bounded by \(e^{-c_2 T}\). Making the analogous computations in (3.14) it follows that

\[ (3.25) \quad \left| K^{-1}_H \left( \int_0^t G(Z) \right) (s) \right|_{L^2([0,T], V)}^2 \leq c_T \left( 1 + |X|_{C([0,T], E)}^2 \right) \leq \tilde{c}_T \left( 1 + |\tilde{Z}|_{C([0,T], E)}^2 \right) \]

where \(\tilde{c}_T \downarrow 0\) as \(T \downarrow 0\) so (3.20) is satisfied. Thus the method in (3.15)–(3.17) can be used to verify (3.21).

Now the existence and the uniqueness of a weak solution of (3.4) is verified for \(H \in (1/2, 1)\).
**Theorem 3.4.** If $H \in (1/2, 1)$, (H1)–(H3) are satisfied and

\[(3.26) \quad \|G(x) - G(y)\| \leq k_G \|x - y\|^\gamma\]

for all $x, y \in E$, some $\gamma \in (0, 1]$, $k_G > 0$ and $\tilde{Z} \in C^\beta([0, T], V)$ for some $\beta$ satisfying

\[(3.27) \quad \beta > \frac{H - 1}{2}\]

where $\tilde{Z}$ is the stochastic convolution process in (H2), then the equation (3.4) has a weak solution. If, additionally, (3.10) is satisfied, then the weak solution is unique.

**Proof.** Initially, the existence of a solution is verified as in the proof of Theorem 3.3. It is shown that

\[(3.28) \quad \int_0^T G(Z(s)) \, ds \in \mathfrak{D}_{0+}^{H + \frac{1}{2}} (L^2([0, T], V)) \quad \text{a.s.}\]

and

\[(3.29) \quad \mathbb{E} \exp[\rho(Z)] = 1\]

where $\rho$ is given by (3.13). By (2.7) it follows that

\[(3.30) \quad \left| K_H^{-1} \left( \int_0^T G(Z) \right) \right|^2 = c_H^{-1} \left| u_{H - 1} D_{0+}^{H - \frac{1}{2}} \left( u_{\frac{1}{2} - H} G(Z) \right) \right|^2_{L^2([0, T], V)}

\begin{align*}
&= c_H^{-1} \int_0^T \left\| \frac{s^{1-H} G(Z(s))}{\Gamma(\frac{1}{2} - H)} \right\|^2_{L^2([0, T], V)} \left( \frac{s^{\frac{1}{2} - H} G(Z(s))}{s^{H - \frac{1}{2}}} \right) ds \\
&\quad + \left( H - \frac{1}{2} \right) \int_0^s \frac{s^{\frac{1}{2} - H} G(Z(s)) - r^{\frac{1}{2} - H} G(Z(r))}{(s - r)^{H + \frac{1}{2}}} \, dr \left| u_{\frac{1}{2} - H} G(Z(r)) \right|^2_{L^2([0, T], V)} ds \\
&\quad + \int_0^s \frac{\|G(Z(s)) - G(Z(r))\|}{(s - r)^{H + \frac{1}{2}}} dr \left| u_{\frac{1}{2} - H} G(Z(r)) \right|^2_{L^2([0, T], V)} ds
\end{align*}

Using (3.9) and (3.26), the analyticity of the semigroup $S(\cdot)$ on $V$ and the inequality

\[
\int_0^s \frac{s^{\frac{1}{2} - H} - r^{\frac{1}{2} - H}}{(s - r)^{H + \frac{1}{2}}} \, dr \leq cs^{1 - 2H}
\]
where $c$ is a generic constant, it follows that

\begin{equation}
(3.31) \quad \left| K_H^{-1} \left( \int_0^T G(Z) \right) \right|_{L^2([0,T],V)} \leq c_T \left( 1 + \|X_0\|_E^2 + |\tilde{Z}|_{C([0,T],E)}^2 \right) \\
+ c_T \int_0^T \left[ \left( \int_0^s \|S(s)X_0 - S(r)X_0\|_E^\gamma \frac{dr}{(s-r)^{H+\frac{1}{2}}} \right)^2 + \left( \int_0^s \|\tilde{Z}(s) - \tilde{Z}(r)\|_E^\gamma \frac{dr}{(s-r)^{H+\frac{1}{2}}} \right)^2 \right] ds \\
\leq c_T \left( 1 + \|X_0\|_E^2 + |\tilde{Z}|_{C([0,T],E)}^2 \right) + c_T \int_0^T \left( \|X_0\|_E \int_0^s \frac{(s-r)^{\gamma\lambda}}{r^{\gamma\lambda}(s-r)^{H+\frac{1}{2}}} dr \right)^2 ds \\
+ c_T \int_0^T |\tilde{Z}|_{C^\beta([0,T],V)}^2 \left( \int_0^s \frac{(s-r)^{\beta\gamma}}{r^{\beta\gamma}(s-r)^{H+\frac{1}{2}}} dr \right)^2 ds
\end{equation}

where $\lambda > 0$ satisfies $\gamma\lambda < 1$ and $H + 1/2 - \gamma\lambda < 1$. It follows that

\begin{equation}
(3.32) \quad \left| K_H^{-1} \left( \int_0^T G(Z) \right) \right|_{L^2([0,T],V)}^2 \leq c_T \left( 1 + \|X_0\|_E^2 + |\tilde{Z}|_{C([0,T],E)}^2 + |\tilde{Z}|_{C^\beta([0,T],V)}^2 \right)
\end{equation}

where $c_T \downarrow 0$ as $T \downarrow 0$ so (3.28) is verified and by the Fernique inequality (3.29) is also verified.

Now the uniqueness of the weak solution is verified. Let $(\tilde{X}(t), t \in [0, T])$ be the solution to the equation (3.4) on a probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P})}$. As in the proof of Theorem 3.3, it is shown that

\begin{equation}
(3.33) \quad \int_0^T G(\tilde{X}) \; ds \in L^2_{0+}([0,T],V) \quad \text{a.s}
\end{equation}

and

\begin{equation}
(3.34) \quad \mathbb{E} \exp \left[ \rho(\tilde{X}) \right] = 1
\end{equation}

where $\rho$ is given by (3.13). It is necessary to obtain the inequality (3.22) that is used in the proof of Theorem 3.3. Note that all of the assumptions of Theorem 3.3 are satisfied here and there is the inequality

\begin{equation}
(3.35) \quad \|\tilde{X}\|_{C^\beta([0,T],V)} \leq L \left( 1 + \|X_0\|_E + |\tilde{Z}|_{C([0,T],E)} + |\tilde{Z}|_{C^\beta([0,T],V)} \right)
\end{equation}

where $\tilde{X}(t) = \tilde{X}(t) - S(t)X_0$ and $L > 0$. Let $w(t) = \tilde{X}(t) - S(t)X_0 - \tilde{Z}(t)$. The process $w$ satisfies

\begin{equation}
(3.36) \quad w(t) = \int_0^t S(t-r)\psi(r) \; dr
\end{equation}
for \( t \in [0, T] \) where
\[
\psi(t) = F(w(t)) + S(t)X_0 + \tilde{Z}(t).
\]

By the inequalities (3.10) and (3.35) it follows that \( \psi \in L^\infty([0, T], V) \) a.s. \( P \). Since the semigroup \( S(\cdot) \) is analytic on \( V \), \( w \) is \( \alpha \)-Hölder continuous for each \( \alpha \in (0, 1) \), and using the method of proof of [26, Theorem 4.3.1] there are constants \( c_i > 0 \) for \( i = 1, 2 \) such that
\[
|w|_{C^\beta([0, T], V)} \leq c_1 |\psi|_{L^\infty([0, T], V)} \leq c_2 \left( |w|_{L^\infty([0, T], E)} + \|X_0\|_E + |\tilde{Z}|_{C([0, T], E)} \right).
\]

Thus
\[
|\tilde{X}|_{C^\beta([0, T], V)} \leq |w|_{C^\beta([0, T], V)} + |\tilde{Z}|_{C^\beta([0, T], V)} \leq c_2 \left( |w|_{L^\infty([0, T], E)} + \|X_0\|_E + |\tilde{Z}|_{C([0, T], E)} + |\tilde{Z}|_{C^\beta([0, T], V)} \right).
\]

Using (3.22) again to bound \( |w|_{L^\infty([0, T], E)} \) the inequality (3.35) follows.

Now, using the methods for the inequalities (3.30)–(3.32), where \( Z(t) \) is replaced by \( \tilde{X}(t) = S(t)X_0 + \tilde{X}(t) \) it follows that
\[
|K_H^{-1}\left( \int_0^t G \right)|^2_{L^2([0, T], V)} \leq c_T \left( 1 + \|X_0\|_E + |\tilde{X}|^2_{C([0, T], V)} + |\tilde{X}|_{C^\beta([0, T], V)} \right)
\]
where \( c_T \downarrow 0 \) as \( T \downarrow 0 \), which by (3.22) and (3.35) verifies (3.33). The equality (3.36) is obtained from (3.39) by the Fernique inequality as in the proof of Theorem 3.3.

**Remark 3.5.** The proofs of Theorem 3.3 and Theorem 4.4 have verified, in addition to weak existence and uniqueness of a solution to (3.4), the mutual absolute continuity (equivalence) of the probability laws of the solution to (3.4) and the solution of (3.4) with \( F \equiv 0 \) (the fractional Ornstein-Uhlenbeck process) in the path space.

The next objective is to relax the linear growth conditions (3.9) and (3.10) and the Hölder continuity (3.26). The linear growth condition is replaced by a dissipativity condition of the drift term of (3.4), but some other conditions are also imposed so that there is existence and (strong) uniqueness of a mild solution. The main contribution of the following two theorems is a mutual absolute continuity of the probability laws of the solutions of (3.4) and (3.4) with \( F \equiv 0 \).

Initially, the case \( H \in (0, 1/2) \) is considered.
Theorem 3.6. Let $H \in (0, 1/2)$ and (H1) and (H2) be satisfied. Let $\Phi \in \mathcal{L}(V)$ be injective, $\Phi^{-1} \in \mathcal{L}(E, V)$ and $(S(t)|_E, t \geq 0)$ be a strongly continuous semigroup on $E$ such that

\begin{equation}
|S(t)|_{\mathcal{L}(E)} \leq e^{\tilde{w}t}
\end{equation}

for $t \geq 0$ and some $\tilde{w} \in \mathbb{R}$. Let $F : E \to E$ be continuous and satisfy

\begin{equation}
\|F(x)\|_E \leq k_1 \left(1 + \|x\|^\rho_E\right)
\end{equation}

for $x \in E$ for some $k_1 \geq 0$ and $\rho \geq 1$ and for each pair $x, y \in E$, there is a $z^* \in \partial \|x - y\|_E$ where $\partial \|z\|_E$ is the subdifferential of the norm $\|\cdot\|_E$ at $z \in E$ such that

\begin{equation}
\langle F(x) - F(y), z^* \rangle_{E, E^*} \leq k_2 \|x - y\|_E
\end{equation}

for some $k_2 \in \mathbb{R}$, that is, $F - k_2 I$ is dissipative on $V$. Then there is one and only one mild solution of (3.4) and its probability law on $\tilde{\Omega} = C([0, T], E)$ is mutually absolutely continuous with respect to the probability law of the fractional Ornstein-Uhlenbeck process (3.24) on $\Omega$.

Proof. Let $(F_\lambda, \lambda > 0)$ be a family of Lipschitz continuous functions from $E$ to $E$ such that each $F_\lambda$ satisfies the inequalities (3.41) and (3.42) for $F$ with the same constants $\rho, k_1, k_2$. It is shown that there is a $\bar{k} > 0$ depending only on $\tilde{w}, k_1,$ and $k_2$ such that

\begin{equation}
\|v_\lambda(t)\|_E \leq \bar{k} \left(1 + \|X_0\|_E + \|\phi\|^\rho_{C([0, T], E)}\right)
\end{equation}

for $t \in [0, T]$ is satisfied for each $\lambda > 0$ and $\phi \in C([0, T], E)$ where $v_\lambda$ is a solution of the equation

\begin{equation}
v_\lambda(t) = S(t)X_0 + \int_0^t S(t - r)F_\lambda(v(r) + \phi(r)) \, dr
\end{equation}

for $t \in [0, T]$.

To verify the inequality (3.43), it can be assumed by translation that $k_2 = 0$ in (3.42) (replace $F_\lambda$ and $A$ by $F_\lambda - k_2 I$ and $A + k_2 I$ respectively). Thus $F_\lambda$ is dissipative on $E$ for each $\lambda > 0$ and by the assumptions

\begin{equation}
\langle A_E z, z^* \rangle_{E, E^*} \leq \tilde{w} \|z\|^2_E
\end{equation}
for each \( z \in \text{Dom}(A_E) \) and \( z^* \in \partial\|z\|_E \) where \( A_E \) is the restriction of \( A \) to \( E \) that generates the semigroup \( S(\cdot)|_E \). For each pair \( x, y \in \text{Dom}(A_E) \) and \( \lambda > 0 \), there is a \( z^* \in \partial\|x - y\|_E \) such that

\[
\langle A_E(x - y) + F_\lambda(x) - F_\lambda(y), z^*_\lambda \rangle_{E,E^*} \leq \hat{w}\|x - y\|_E.
\]

By [5, Proposition 5.5.6], there is a sequence \((v^n_\lambda, n \in \mathbb{N})\) such that \( v^n_\lambda \in C^1([0, T], E) \cap C([0, T], \text{Dom}(A_E)) \) such that \( v^n_\lambda \rightarrow v_\lambda \) and \( \delta^n_\lambda = \frac{d}{dt}v^n_\lambda - A_Ev^n_\lambda - F_\lambda(v^n_\lambda + \phi) \rightarrow 0 \) in \( C([0, T], E) \) as \( n \rightarrow \infty \). It follows that

\[
(3.46) \quad \frac{d}{dt}\|v^n_\lambda(t)\|_E \leq \langle A_Ev^n_\lambda(t) + F_\lambda(v^n_\lambda(t) + \phi(t)), v^n_\lambda(t) \rangle_{E,E^*} + \|\delta^n_\lambda(t)\| + \|\delta^n_\lambda(t)\|_E
\]

\[
= \langle A_Ev^n_\lambda(t) + F_\lambda(v^n_\lambda(t) + \phi(t)) - F_\lambda(\phi(t)), v^n_\lambda(t) \rangle_{E,E^*} + \langle F_\lambda(\phi(t)), v^n_\lambda(t) \rangle_{E,E^*} + \|\delta^n_\lambda(t)\|_E
\]

\[
\leq \hat{w}\|v^n_\lambda(t)\|_E + k_2 \left( 1 + |\phi|_C^{\rho}[0, T], E \right) + \|\delta^n_\lambda(t)\|_E
\]

for \( t \in [0, T] \). Using the Gronwall Lemma, and letting \( n \rightarrow \infty \), verifies the inequality (3.43).

The mild solution to (3.4) can be expressed as \( X(t) = v(t) + \tilde{Z}(t) \) where \( v \) satisfies the equation

\[
(3.47) \quad v(t) = S(t)X_0 + \int_0^t S(t - r)F(v(r) + \tilde{Z}(r)) \, dr
\]

for \( t \in [0, T] \). Thus the existence and the uniqueness of a mild solution follows from the corresponding pathwise deterministic result (cf., [5, Proposition 5.5.6]).

The equivalence of the probability laws is shown by application of Theorem 3.1. As in the proof of Theorem 3.3, it suffices to show that

\[
(3.48) \quad \int_0^t G(Z(s)) \, ds \in L_{0+}^{1-H+\frac{1}{2}}(L^2([0, T], V))
\]

and

\[
(3.49) \quad \mathbb{E} \exp[\rho(Z)] = 1
\]

where \( \rho \) is given by (3.13). While \( G \) is not assumed to have at most linear growth as in Theorem 3.3, there is the growth condition

\[
(3.50) \quad \|G(x)\| \leq \hat{k} \left( 1 + \|x\|_E^\rho \right)
\]
for all $x \in E$ and a constant $\hat{k}$. Proceeding as in (3.14), it follows that

$$
\left|K^{-1}_H \left( \int_0^T G(Z) \right) \right|^2_{L^2([0,T], V)} \leq c_1 \int_0^T \left( s^{H-\frac{1}{2}} \left\| \int_0^s r^{\frac{H}{2}-H} (s-r)^{-\frac{1}{2}} H G(Z(r)) \, dr \right\| \right)^2 \, ds
$$

$$
\leq c_2 \left( 1 + |\tilde{Z}|_C^{\rho} + \sup_{t \in [0,T]} \|S(t)X_0\|_E^{\rho} \right) \int_0^T s^{2H-1} \left( \int_0^s r^{\frac{H}{2}-H} (s-r)^{-\frac{1}{2}} H \, dr \right)^2 \, ds
$$

$$
\leq c_3 \left( 1 + \|X_0\|_E^{2\rho} + |\tilde{Z}|_C^{2\rho} \right)
$$

(3.51)

for suitable constants $c_1, c_2, c_3$. This inequality verifies (3.48). To verify the equality (3.49), it suffices to assume that $F$ is dissipative (that is, $k_2 = 0$ in (3.42)). Since $F$ is continuous, it is $m$-dissipative (cf., [20]) so the family $(F_\lambda, \lambda > 0)$ of Yosida approximations of $F$ is defined as

$$
F_\lambda(x) = F\left( R_\lambda(x) \right) = \frac{1}{\lambda} (R_\lambda(x) - x)
$$

for $x \in E$ where

$$
R_\lambda(x) = (I - \lambda F)^{-1}(x).
$$

(3.52)

It is well known that $F_\lambda : E \to E$ for $\lambda > 0$ is Lipschitz continuous, so by Theorem 3.3, there is the equality

$$
E \exp \left[ \rho_\lambda(Z) \right] = 1
$$

(3.54)

for $\lambda > 0$ where

$$
\rho_\lambda(Z) = \int_0^T \left\langle K^{-1}_H \left( \int_0^s G_\lambda(Z) \right) (t), dW(t) \right\rangle - \frac{1}{2} \int_0^T \left\| K^{-1}_H \left( \int_0^s G_\lambda(Z) \right) (t) \right\|^2 \, dt
$$

(3.55)

and $G_\lambda := \Phi^{-1}F_\lambda$. As in (3.51), it follows that

$$
E \left| K^{-1}_H \left( \int_0^T (G_\lambda(Z) - G(Z)) \right) \right|_{L^2([0,T], V)} \leq c_T \int_0^T \left( s^{H-\frac{1}{2}} \int_0^s r^{\frac{H}{2}-H} (s-r)^{-\frac{1}{2}} H \|G_\lambda(Z(r)) - G(Z(r))\| \, dr \right)^2 \, ds.
$$

(3.56)
By some well known properties of the Yosida approximations and for \( x \in E \),

\[
\|G_\lambda(x) - G(x)\| \leq |\Phi^{-1}|_{\mathcal{L}(E,V)} \|F_\lambda(x) - F(x)\|
\]

it follows that \( F_\lambda \to F \) as \( \lambda \to 0 \) and the right hand side of (3.57) tends to zero as \( \lambda \downarrow 0 \), and

\[
\|G_\lambda(x)\| \leq |\Phi^{-1}|_{\mathcal{L}(E,V)} \|F_\lambda(x)\|_E
\]

\[
\leq |\Phi^{-1}|_{\mathcal{L}(E,V)} \|F(x)\|_E
\]

\[
\leq |\Phi^{-1}|_{\mathcal{L}(E,V)} k_1 (1 + \|x\|_E^p)
\]

so the right-hand side of (3.56) tends to zero as \( \lambda \downarrow 0 \). For a sequence \( (\lambda_n, n \in \mathbb{N}) \) that decreases to zero it follows that

\[
\lim_{n \to \infty} \exp \left[ \rho_{\lambda_n}(Z) \right] = \exp \left[ \rho(Z) \right] \quad \text{a.s.} \quad \mathbb{P}.
\]

To obtain the equality (3.54) from the equality (3.59) for \( \lambda_n, n \in \mathbb{N} \), it is necessary and sufficient to show that the sequence \( \left( \exp \left[ \rho_{\lambda_n}(Z) \right], n \in \mathbb{N} \right) \) is uniformly integrable. A sufficient condition for this uniform integrability is to verify that

\[
\sup_n \mathbb{E} \left[ \left( \exp \left[ \rho_{\lambda_n}(Z) \right] \right) \log \left( \exp \left[ \rho_{\lambda_n}(Z) \right] \right) \right] = \sup_n \mathbb{E} \left[ \left( \exp \left[ \rho_{\lambda_n}(Z) \right] \right) \rho_{\lambda_n}(Z) \right] < \infty.
\]

By Theorem 3.3,

\[
\mathbb{E} \left[ \rho_{\lambda_n}(Z) \exp \left[ \rho_{\lambda_n}(Z) \right] \right] \leq \mathbb{E}_{\lambda_n} \left[ 2 \int_0^T \left\| \mathcal{K}_H^{-1} \left( \int_0^t G_{\lambda_n}(Z) \right) \right\|^2 dt \right]
\]

where \( \mathbb{E}_{\lambda_n} \) is expectation with respect to \( \mathbb{P}_{\lambda_n} \) and

\[
\frac{d\mathbb{P}_{\lambda_n}}{d\mathbb{P}} = \exp \left[ \rho_{\lambda_n}(Z) \right]
\]

and \( Z(\cdot) \) satisfies the equation (2.27). On the probability space with the measure \( \mathbb{P}_{\lambda_n} \), \( Z(\cdot) \) satisfies the following semilinear equation where \( B(\cdot) \) is a fractional Brownian motion with respect to \( \mathbb{P}_{\lambda_n} \)

\[
dX_{\lambda_n}(t) = (AX(t) + F_{\lambda_n}(X(t))) \, dt + \Phi dB(t)
\]

\[
X_{\lambda_n}(0) = X_0.
\]
Since $F_{\lambda_n}$ is Lipschitz continuous, there is a unique mild solution on a given probability space so it suffices to show

$$\mathbb{E} \int_0^T \left\| K_H^{-1} \left( \int_0^t G_{\lambda_n}(X_{\lambda_n}(s)) \right)(t) \right\|^2 dt \leq c$$

for some $c \in \mathbb{R}^+$ that does not depend on $\lambda_n$. Repeating the inequalities (3.51) where $G$ and $Z$ are replaced by $G_{\lambda_n}$ and $X_{\lambda_n}$ respectively and using the inequality (3.58) it follows that

$$\int_0^T \left\| K_H^{-1} \left( \int_0^t G_{\lambda_n}(X_{\lambda_n}(s)) \right)(t) \right\|^2 dt \leq c_5 \left( 1 + \|X_0\|^2_E + \|\tilde{X}_{\lambda_n}\|_{C([0, T], E)}^{2p} \right)$$

for a constant $c_k$ that does not depend on $n \in \mathbb{N}$ where $\tilde{X}_{\lambda_n}(t) = X_{\lambda_n}(t) - S(t)X_0$. By the inequality (3.43) there is a constant $c_6$ that does not depend on $n$ such that

$$\mathbb{E} \int_0^T \left\| K_H^{-1} \left( \int_0^t G_{\lambda_n}(X_{\lambda_n}(s)) \right)(t) \right\|^2 dt \leq c_6 \left( 1 + \|X_0\|^2_E + \mathbb{E}\|Z\|_{C([0, T], E)}^{4p^2} \right) = C < \infty.$$  

This inequality verifies (3.60). Thus the sequence $(\exp \left[ t\rho_{\lambda_n}(Z) \right], n \in \mathbb{N})$ converges in $L^1$ and the equality (3.54) is satisfied. \(\square\)

Now the case $H \in (1/2, 1)$ is considered.

**Theorem 3.7.** Let $H \in (1/2, 1)$ and the other assumptions in Theorem 3.6 be satisfied. Let $\Phi^{-1} \in \mathcal{L}(V)$, $\tilde{Z} \in C^\beta([0, T], V)$ for some $\beta \in (0, 1)$,

$$\langle F(x) - F(y), x - y \rangle \leq k_2 \|x - y\|$$

for each pair $x, y \in E$ and a $k_2 \in \mathbb{R}_+$ (that is, $F - k_2I$ is dissipative on $E$ with respect to the norm on $V$) and

$$\|F(x) - F(y)\| \leq k_3 \left( 1 + \|x\|_E^q + \|y\|_E^q \right) \|x - y\|^\gamma$$

for each $x, y \in E$, with some $k_3 > 0$, $q \geq 1$, and $\gamma \in (0, 1]$ such that

$$\gamma \beta > H - \frac{1}{2}.$$  

Then there is one and only one mild solution to (3.4) and its probability law is mutually absolutely continuous with respect to the probability law of the fractional Ornstein-Uhlenbeck process (2.27) on $\tilde{\Omega}$. 

Proof. As in the proof of Theorem 3.6, it is shown that

\begin{equation}
\int_0^t G(Z(s)) \, ds \in L^1_{0+} \left( L^2([0, T], V) \right)
\end{equation}

and

\begin{equation}
\mathbb{E} \exp [\rho(Z)] = 1.
\end{equation}

The methods to verify (3.69) and (3.70) are similar to those used in the proof of Theorem 3.6, but now the operator $K^{-1}_H$ has a different form. Using the inequality (3.41) and the Hölder continuity condition (3.67) it follows that

\begin{equation}
\left| K^{-1}_H \left( \int_0^t G(Z) \right) \right|^2_{L^2([0, T], V)} \leq c_1 \int_0^T \left( s^{1-H} \| G(Z(s)) \| + s^{H-1} \int_0^s \frac{r^{1-H} - s^{1-H} + s^{1-H}}{(s-r)^{H+\frac{1}{2}}} \| G(Z(r)) \| \, dr \right. \\
\left. + \int_0^s \frac{\| G(Z(s)) - G(Z(r)) \|}{(s-r)^{H+\frac{1}{2}}} \, dr \right) ds \\
\leq c_2 \left[ 1 + |Z|^{2p}_{C([0, T], E)} + c_3 \left( 1 + |Z|^{2q}_{C([0, T], E)} \right) \right. \\
\left. \cdot \left( \int_0^T \int_0^s \left\| S(s) X_0 - S(r) X_0 \right\|^\gamma + \left\| \bar{Z}(s) - \bar{Z}(r) \right\|^{\frac{\gamma}{2}} \, dr \right)^{\frac{2}{\gamma}} \right] ds
\end{equation}

for some constants $c_1, c_2, c_3$. By the analyticity of the semigroup $S(\cdot)$ on $V$, it follows that

\begin{equation}
\left| K^{-1}_H \left( \int_0^t G(Z) \right) \right|^2_{L^2([0, T], V)} \leq c_4 \left[ 1 + \| X_0 \|^{2p}_E \right. \\
\left. + \left( \| X_0 \|^{2q}_{C([0, T], V)} + |Z|^{2q}_{C([0, T], V)} \right) \left( \| X_0 \|^{2\gamma}_E + |Z|^{2\gamma}_{C^{\beta}([0, T], V)} \right) \right] \\
\leq c_5 \left( 1 + \| X_0 \|^{m}_E + |Z|^{m}_{C([0, T], V)} + |Z|^{m}_{C^{\beta}([0, T], V)} \right)
\end{equation}

for some constants $c_4$ and $c_5$ and $m$ sufficiently large. Thus (3.69) is verified. To verify the equality (3.70) consider the family of Yosida approximations $(F_{\lambda}, \lambda > 0)$ of $F$ as in the proof of Theorem 3.6. By the dissipativity of $F$ in the norm on $V$, $F_{\lambda} : V \to V$ is Lipschitz continuous for each $\lambda > 0$ and has at most polynomial growth so $F_{\lambda}$ satisfies the assumptions of Theorem 3.4 so that

\begin{equation}
\mathbb{E} \exp [\rho_{\lambda}(Z)] = 1
\end{equation}
where $\rho_\lambda$ is given by (3.55). By the method to obtain the inequality (3.71) it follows that

$$
E \left| K^{-1}_H \left( \int_0^T (G_\lambda(Z) - G(Z)) \right) \right|^2_{L^2([0,T],V)} 
\leq c_6 E \int_0^T \left[ s^{\frac{1}{2}-H} \|G_\lambda(Z(s)) - G(Z(s))\| + s^{H-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}}{(s-r)^{H+\frac{1}{2}}} \|G_\lambda(Z(r)) - G(Z(r))\| \, dr 
+ \int_0^s \frac{\|G_\lambda(Z(s)) - G(Z(s)) - G_\lambda(Z(r)) + G(Z(r))\|}{(s-r)^{H+\frac{1}{2}}} \, dr \right]^2 \, ds.
$$

By the inequalities (3.57) and (3.58), it follows that $\|G_\lambda(x) - G(x)\| \to 0$ as $\lambda \downarrow 0$ for each $x \in E$ and the family $(G_\lambda, \lambda > 0)$ satisfies the growth condition

$$
\|G_\lambda(x)\| \leq c_7 \left(1 + \|x\|^p_E\right)
$$
for $x \in E$ and some $c_7 > 0$. From the $V$-dissipativity of $F$, it follows by [5, Proposition 5.5.3] that

$$
\|R_\lambda(x) - R_\lambda(y)\| \leq \|x - y\|
$$
for $x,y \in E$ so that

$$
\|F_\lambda(x) - F_\lambda(y)\| = \|F(R_\lambda(x)) - F(R_\lambda(y))\| \leq k_3 \left(1 + \|R_\lambda(x)\|^q_E + \|R_\lambda(y)\|^q_E\right) \|x - y\|^\gamma
$$
for $x,y \in E$. Since

$$
\|R_\lambda(x)\|_E \leq \|x\|_E + \lambda \|F(x)\|_E \leq c_8 \left(1 + \|x\|^p_E\right)
$$
for $x \in E$, $c_8 \in \mathbb{R}_+$, and $\lambda \in (0,1]$, there is the inequality

$$
\|F_\lambda(x) - F_\lambda(y)\| \leq c_9 \left(1 + \|x\|^m_E + \|y\|^m_E\right) \|x - y\|^\gamma
$$
for $x,y \in E$, $c_9 \in \mathbb{R}_+$, $m \geq 1$, and $\lambda \in (0,1]$. So $F_\lambda$ and $G_\lambda$ satisfy the inequality (3.67) uniformly in $\lambda \in (0,1]$. Thus the right hand side of the inequality (3.74) tends to zero as $\lambda \downarrow 0$ by the Dominated Convergence Theorem where a majorizing function is provided by the estimates (3.75) and (3.77) whose integrability is shown as in (3.71) and (3.72) and there is a decreasing sequence $(\lambda_n, n \in \mathbb{N})$ whose limit is zero such that

$$
\lim_{n \to \infty} \exp \left[ \rho_{\lambda_n}(Z) \right] = \exp \left[ \rho(Z) \right] \quad \text{a.s. } P.
$$
The uniform integrability of the sequence \( \left( \exp \left[ \rho_{\lambda_n}(Z) \right], n \in \mathbb{N} \right) \) is shown by verifying the analogue of (3.60). Equivalently,

\[
(3.79) \quad \sup_n E \int_0^T \left\| K_H^{-1} \left( \int_0^t G_{\lambda_n} (X_{\lambda_n}) \right) (t) \right\|^2 dt \leq c < \infty
\]

where \( X_{\lambda_n} (\cdot) \) is the unique mild solution to the equation (3.62). The analogous inequalities (3.71)–(3.74) are obtained by replacing \( G \) by \( G_{\lambda} \) using the polynomial growth bound and the local Hölder continuity that are uniform in \( (\lambda_n, n \in \mathbb{N}) \) and \( Z(\cdot) \) is replaced by \( X_{\lambda_n} (\cdot) \). For some constants \( c_{10} \) and \( m \geq 1 \),

\[
(3.80) \quad \left\| K_H^{-1} \left( \int_0^t G_{\lambda_n} (X_{\lambda_n}) \right) \right\|^2_{L^2([0,T], V)} \leq c_{10} \left( 1 + \| X_0 \|^m_E + | \tilde{X}_{\lambda_n} |^m_{C([0,T], E)} + | \tilde{X}_{\lambda_n} |^m_{C^\rho([0,T], V)} \right)
\]

where \( \tilde{X}_{\lambda_n} (t) = X_{\lambda_n} (t) - S(t)X_0 \). By the inequality (3.43), it follows that

\[
(3.81) \quad | \tilde{X}_{\lambda_n} |_{C([0,T], E)} \leq c_{11} \left( 1 + \| X_0 \| + | Z |^p_{C([0,T], E)} \right)
\]

for some \( c_{11} > 0 \). Let \( w_{\lambda_n} (t) = \tilde{X}_{\lambda_n} (t) - Z(t) \) so that

\[
(3.82) \quad w_{\lambda_n} (t) = \int_0^t S(t-r) F_{\lambda_n} \left( w_{\lambda_n} (s) + S(s)X_0 + \tilde{Z}(s) \right) ds
\]

for \( t \in [0,T] \). The inequality (3.81) provides a uniform bound on \( | w_{\lambda_n} |_{C([0,T], E)} \) so by repeating the arguments for the inequalities (3.37) and (3.38), it follows that

\[
(3.83) \quad | X_{\lambda_n} |_{C^\rho([0,T], V)} \leq c_{12} \left( 1 + \| X_0 \| + | Z |^p_{C([0,T], E)} + | \tilde{Z} |^p_{C^\rho([0,T], V)} \right)
\]

for some \( c_{12} > 0 \). The inequalities (3.80) and (3.81) verify the inequality (3.79) so the sequence \( \left( \exp \left[ \rho_{\lambda_n}(Z) \right], n \in \mathbb{N} \right) \) is uniformly integrable and the equality (3.73) is verified. \( \square \)

4. SOME EXAMPLES

The first example is a finite dimensional stochastic equation with a nonlinear drift. Consider the equation

\[
(4.1) \quad dX(t) = f(X(t)) dt + \Phi dB(t)
\]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( \Phi \in \mathcal{L}(\mathbb{R}^n) \) and \((B(t), t \geq 0)\) is an \( \mathbb{R}^n \)-valued standard fractional Brownian motion with Hurst parameter \( H \in (0,1) \). This case can be subsumed in the infinite dimensional
results given here though some of the assumptions and the results simplify significantly. Let \( E = V = \mathbb{R}^n \), \( S(t) = I \) for \( t \in \mathbb{R}_+ \) and assume that \( Q = \Phi \Phi^* \) is positive definite. The process
\[
\left( \int_0^t \Phi dB, t \in [0, T] \right)
\]
has sample paths in \( C^\beta ([0, T], \mathbb{R}^n) \) for \( 0 < \beta < H \). If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is Borel measurable and
\[
\| f(x) \| \leq k_1 (1 + \| x \|)
\]
for some \( k_1 > 0 \) and all \( x \in \mathbb{R}^n \) then for \( H < \frac{1}{2} \) there is one and only one weak solution of (4.1) by Theorem 3.3. If, additionally, it is assumed that
\[
\| f(x) - f(y) \| \leq k \| x - y \|^\gamma
\]
for all \( x, y \in \mathbb{R}^n \) and some \( \gamma > 1 - \frac{1}{2H} \), then for \( H > \frac{1}{2} \), there is one and only one weak solution. In each of these cases, the probability measure of the solution is mutually absolutely continuous with respect to the probability measure of the process \( (\Phi B(t), t \in [0, T]) \).

Now, replace the inequality in (4.2) by
\[
\| f(x) \| \leq k_1 (1 + \| x \|^p)
\]
for some \( p \geq 1 \) and \( k_1 > 0 \). Assume that \( f : \mathbb{R}^n \to \mathbb{R}^n \) is continuous and satisfies
\[
\langle f(x) - f(y), x - y \rangle \leq k_3 \| x - y \|^2
\]
for some \( k_3 > 0 \) and all \( x, y \in \mathbb{R}^n \). If \( H > \frac{1}{2} \), then assume that
\[
\| f(x) - f(y) \| \leq k_4 (1 + \| x \|^q + \| y \|^q) \| x - y \|^\gamma
\]
for some \( q \geq 1 \), \( k_4 > 0 \), \( \gamma > 1 - \frac{1}{2H} \). Then Theorem 3.6 can be used to verify that the probability law of the solution of (4.1) is mutually absolutely continuous with respect to the probability law of \( (\Phi B(t), t \in [0, T]) \). Furthermore, there is one and only one weak solution of (4.1). Note that the inequalities (4.4)—(4.6) are satisfied for the important case of models where \( f \) is a polynomial of odd degree with a negative leading coefficient.
The second example is a linear stochastic partial differential equation. The solution is obtained from the results in [25]. Consider a $2m$th order stochastic parabolic equation

$$
\frac{\partial u}{\partial t}(t, \xi) = [L_{2m}u](t, \xi) + \eta(t, \xi)
$$

for $(t, \xi) \in [0, T] \times \vartheta$ with the initial condition

$$
u(0, \xi) = x(\xi)
$$

for $\xi \in \vartheta$ and the Dirichlet boundary condition

$$
\frac{\partial^k u}{\partial v^k}(t, \xi) = 0
$$

for $(t, \xi) \in [0, T] \times \partial \vartheta$, $k \in \{0, \ldots, m-1\}$, $\frac{\partial}{\partial v}$ denotes the conormal derivative, $\vartheta$ is a bounded domain in $\mathbb{R}^d$ with a smooth boundary and $L_{2m}$ is a $2m$th order uniformly elliptic operator

$$
L_{2m} = \sum_{|\alpha| \leq 2m} a_\alpha(\xi) D^\alpha
$$

and $a_\alpha \in C_b^\infty(\vartheta)$. For example, if $m = 1$ then this equation is called the stochastic heat equation. The process $\eta$ denotes a space dependent noise process that is fractional in time with the Hurst parameter $H \in (0, 1)$ and, possibly, in space. The system (4.7)–(4.9) is modeled as

$$
dZ(t) = AZ(t) dt + \Phi dB(t)
$$

$$
Z(0) = x
$$

in the space $V = L^2(\vartheta)$ where $A = L_{2m}$.

$$
\text{Dom}(A) = \left\{ \varphi \in H^{2m}(\vartheta) \mid \frac{\partial^k \varphi}{\partial v^k} = 0 \text{ on } \partial \vartheta \text{ for } k \in \{0, \ldots, m-1\} \right\},
$$

$\Phi \in \mathcal{L}(V)$ defines the space correlation of the noise process and $(B(t), t \geq 0)$ is a cylindrical standard fractional Brownian motion in $V$. For $\Phi = I$, the noise process is uncorrelated in space. It is well known that $A$ generates an analytic semigroup $(S(t), t \geq 0)$. Furthermore

$$
|S(t)\Phi|_{\mathcal{L}^2(V)} \leq |S(t)|_{\mathcal{L}^2(V)} |\Phi|_{\mathcal{L}^2(V)} \leq ct^{-\frac{d}{4m}}
$$

for $t \in [0, T]$. It is assumed that there is a $\delta_1 > 0$ and $\hat{\beta} \in \mathbb{R}$ such that

$$
\text{Im}(\Phi) \subset \text{Dom}\left((\hat{\beta} I - A)^{\delta_1}\right)
$$
so that for \( r \geq 0 \)

\[
(4.14) \quad |S(t)\Phi|_{\mathcal{L}_2(V)} \leq |S(t)(\hat{\beta}I - A)^r|_{\mathcal{L}_2(V)} |(\hat{\beta}I - A)^{-r-\delta_1}|_{\mathcal{L}_2(V)} |(\hat{\beta}I - A)^{\delta_1}\Phi|_{\mathcal{L}_2(V)} \leq ct^{-r}
\]

for \( t \in (0, T] \) assuming that the operator \((\hat{\beta}I - A)^{-r-\delta_1}\) is a Hilbert-Schmidt operator on \( V \), which occurs if

\[
(4.15) \quad r + \delta_1 > \frac{d}{4m}.
\]

For the spaces \( V_\delta = \text{Dom}\left((\hat{\beta}I - A)^\delta\right), \delta \geq 0 \), if

\[
(4.16) \quad H > \frac{d}{2m},
\]

then for any \( \Phi \in \mathcal{L}(V) \), the stochastic convolution process

\[
\left( \int_0^t S(t-r)\Phi dB(r), t \in [0, T] \right)
\]

is well-defined and has a version with \( C^\alpha([0, T], V_\delta) \) sample paths for \( \alpha \geq 0, \delta \geq 0 \) satisfying

\[
(4.17) \quad \alpha + \delta < H - \frac{d}{4m}.
\]

If moreover (4.13) is satisfied, then the previous conclusion is satisfied if

\[
(4.18) \quad \alpha + \delta < H - \frac{d}{4m} + \delta_1.
\]

Note that these results are analogous to the case of a standard Wiener process \( (H = \frac{1}{2}) \) (4).

The third example is a semilinear stochastic heat equation. Consider the equation

\[
(4.19) \quad \frac{\partial y}{\partial t}(t, \xi) = \frac{\partial^2 y}{\partial \xi^2}(t, \xi) + f(y(t, \xi)) + \eta(t, \xi)
\]

for \((t, \xi) \in (0, T) \times (0, 1)\) and

\[
y(0, \xi) = x_0(\xi) \\
\frac{\partial y}{\partial \xi}(t, 0) = \frac{\partial y}{\partial \xi}(t, 1) = 0
\]
for \((t, \xi) \in (0, T) \times (0, 1)\) where \(f : \mathbb{R} \to \mathbb{R}\) and \((\eta(t, \xi), t \in [0, T])\) is a space dependent noise that is fractional in time. The equation is rewritten in a standard infinite dimensional form letting \(V = L^2([0, 1]), A = \frac{\partial^2}{\partial \xi^2}\),

\[
\text{Dom}(A) = \left\{ \phi \in H^2([0, 1]) \mid \frac{\partial}{\partial \xi} \phi(0) = \frac{\partial}{\partial \xi} \phi(1) = 0 \right\}
\]

and \(F : V \to V\) where \(F(x)(\xi) = f(x(\xi))\) for \(x \in V\) and \(\xi \in (0, 1)\). It is assumed that \(Q = \Phi \Phi'\) has a bounded inverse in \(L(V)\).

Initially, consider \(H \in (1/2, 1)\) and let \(f\) satisfy (4.5)–(4.7) with \(\mathbb{R}^n = \mathbb{R}\) (for example, (4.6) is \((f(x) - f(y)) \text{sgn}(x - y) \leq k_3(x - y)\) for \(x, y \in \mathbb{R}\)). By a well known characterization of the subdifferential of the norm on \(E = C([0, 1])\) (cf., [29]), it follows that (3.42), the inequality (3.41), and the assumptions (3.66) and (3.67) of Theorem 3.7 are satisfied with the same \(\gamma\). By the Sobolev Embedding Theorem and the second example, the process \((Z(t), t \in [0, T])\) has \(C([0, T], E) \cap C^\beta([0, T], V)\) sample paths for \(0 < \beta < H - \frac{1}{4}\). This inequality for \(\beta\) yields the inequality \(\gamma > \frac{H - \frac{1}{2}}{H - \frac{1}{4}}\), which can be viewed as a condition on the Hölder exponent of the nonlinear term \(f\). Thus Theorem 3.7 is applicable.

If \(H \in (0, 1/2)\), then the sample paths of \(Z\) are not sufficiently smooth to choose \(E = C([0, T])\). However, if \(E = V\) and \(f : \mathbb{R} \to \mathbb{R}\) is Borel measurable and satisfies the inequality (4.5) with \(\rho = 1\) and \(\mathbb{R}^n = \mathbb{R}\), then there is one and only one weak solution by Theorem 3.6, (4.16), and (4.17), provided \(H > \frac{1}{4}\).

**REFERENCES**


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