A RANDOM WALK AROUND SOME PROBLEMS IN IDENTIFICATION AND STOCHASTIC ADAPTIVE CONTROL

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Abstract. The importance of identification and adaptive control of continuous-time stochastic systems are discussed. Recent developments of stochastic calculus of fractional Brownian motion and its applications to stochastic differential equations are presented. Applications of presented stochastic theory to epilepsy are shown. The educational aspects of stochastic analysis and control are discussed.

Key Words. Identification, Adaptive Control, Fractional Brownian motion, Epilepsy.

1. INTRODUCTION AND MOTIVATION

1.1. Stochastic Analysis and Stochastic Control

Stochastic analysis and stochastic control has rapidly entered many fields including engineering, science, medical sciences, and social sciences. The development of analytical and numerical methods based on PDEs have increased the relevance in everyday life.

- System’s behavior depends on the parameters and the fact that the value of the parameters is unknown makes the system unknown.
- Some crucial information concerning the system is not available to the controller and this information should be learned during the system’s performance.
- The described problem is the problem of adaptive control.

The general approach to adaptive control that is described here exhibits a splitting or separation of identification and adaptive control.

1.2. Identification

The estimators used here for identification include maximum likelihood, least squares, and weighted least squares.

For some cases, the weighted least squares estimator is strongly consistent while the least squares estimator is not.

Some important issues for identification are strong consistency, recursivity, rate of convergence, and asymptotic behavior of estimators.

1.3. Adaptive Control

The adaptive control constructed by the so called certainty equivalence principle, that is the optimal stationary control, is computed by replacing the unknown parameter valued by the current estimates of these values.

Some of the most important issues for the adaptive control include:

- Self-tuning property
  Asymptotically the adaptive control using the estimate of the unknown parameter is as good as the optimal control if we knew the system (the optimal stationary controls as continuous functions of the unknown parameters).
- Self-optimizing property
  The family of average costs converges to the optimal average costs.
- Numerical computations for adaptive control

1.4. Focus on Identification and Adaptive Control of Continuous-Time Stochastic Systems

Many models evolve in time. It is important for the study of discrete time models when the sampling rates are large and for the analysis of numerical round-off errors. Stochastic calculus provides powerful tools: stochastic integral, Itô’s differential, martingales.
1.5. Stochastic Adaptive Control Problems as Applications of the Stochastic Control Theory

We use the certainty equivalence control as an adaptive control, so we need the optimal control given explicitly or the nearly optimal control. Some stochastic adaptive control problems have been solved in cases when
- the parameters are constants
- the parameters are functions of time
- the parameters are random
- the parameters are stochastic processes.

1.6. Weighted Least Squares and Continuous Time Adaptive LQC Control

- The linear Gaussian control problem with ergodic, quadratic cost functional is probably most well known ergodic control problem.
- It is a basic problem to solve for stochastic adaptive control since the optimal control can be easily computed and the existence of an invariant measure follows directly from the stability of the optimal system.
- The problem is solved using only the natural assumptions of controllability and observability.
- The weighted least squares scheme is used to obtain the convergence of the family of estimates (self convergence).
- The scheme is modified by a random regularization to obtain the uniform controllability and observability of the family of estimates.
- A diminishing excitation white noise is used to obtain strong consistency.
- The excitation is sufficient to include the identification of unknown deterministic linear systems.
- The approach eliminates some other assumptions that have previously been used that are unnecessary for the control problem for a known system and are often difficult to verify.
- The approach eliminates the need for random switching or resetting which often occur in previous work.

1.7. Weighted Least Squares Identification

Let \( X(t), t \geq 0 \) be the process that satisfies the stochastic differential equation

\[
dX(t) = AX(t)dt + BU(t)dt + DdW(t)
\]

or

\[
dX(t) = \Theta^T \varphi(t)dt + DdW(t)
\]

where

\[
\Theta^T = [A, B], \quad \varphi(t) = \begin{bmatrix} X(t) \\ U(t) \end{bmatrix},
\]

\( X(0) = X_0, X(t) \in \mathbb{R}^n, U(t) \in \mathbb{R}^m, (W(t), t \geq 0) \) is an \( \mathbb{R}^p \)-valued standard Wiener process, and \( (U(t), t \geq 0) \) is a control from a family that is specified.

The random variables are defined on a fixed complete probability space \((\Omega, \mathcal{F}, P)\) and there is a filtration \((\mathcal{F}_t, t \geq 0)\) defined on this space. It is assumed that \( A \) and \( B \) are unknown.

A family of weighted least squares (WLS) estimates \((\Theta(t), t \geq 0)\) is given by

\[
d\hat{\Theta}(t) = a(t)P(t)\varphi(t)(dX^T(t) - \varphi(t)\hat{\Theta}(t)dt)
\]

\[
dP(t) = -a(t)P(t)\varphi(t)\varphi^T(t)P(t)dt,
\]

where \( \Theta(0) = P(0) > 0 \) are arbitrary,

\[
a(t) = \frac{1}{f(\tau(t))}
\]

with

\[
\tau(t) = e + \int_{t}^{1} |\varphi(s)|^2 ds
\]

and

\[
f \in F = \left\{ f : \mathbb{R}_t \to \mathbb{R}_t, f \text{ is slowly increasing, } \int_{c}^{\infty} \frac{dx}{xf(x)} < \infty \text{ for some } c \geq 0 \right\}.
\]

The following ergodic functional is used

\[
J(U) = \lim_{T \to \infty} \sup_T \int_{0}^{T} \left[ X^T(t)Q_1 X(t) + U^T(t)Q_2 U(t) \right] dt
\]

where \( (U(t), t \geq 0) \) is an admissible control, \( Q_1 \geq 0, Q > 0 \), and the following assumptions are made:

- \((A, B)\) is controllable
- \((A, Q_1^{1/2})\) is observable.

Adaptive Control:

The diminishing excited lagged certainty equivalence control is used.

Identification:

To obtain the strong consistency for the family of estimates, a diminishing excitation is added to the adaptive control.

The complete solution to the adaptive control problem is obtained with the most natural assumptions.

Solutions to the adaptive control problem for stochastic continuous time linear and some nonlinear systems were obtained in \([8, 9]\).

In the recent paper on “Solutions of Stochastic Linear Distributed Parameter Equations with Multiplicative Fractional Gaussian Noise” by T. Duncan, B. Maslowski, and B. Pasik-Duncan, explicit solutions are given for a family of stochastic linear distributed parameter equations with a multiplicative fractional Gaussian noise. The solutions can be strong, weak or mild depending on the particular assumptions on the equation description. A fractional Gaussian noise is
the formal derivative of a fractional Brownian motion which is determined in probability law by its Hurst parameter. In this paper, the Hurst parameter is restricted to the interval (1/2, 1). Only a limited number of results are available for the existence of solutions of stochastic differential equations with a fractional Brownian motion. The solutions given here require the use of a stochastic calculus for a fractional Brownian motion.

The results for existence and uniqueness of solutions for stochastic differential equations with a fractional Brownian motion are incomplete, so it is necessary to consider special classes of stochastic differential equations. A stochastic equation where the diffusion coefficient is deterministic can be solved from the results for the corresponding deterministic equation. However, a stochastic equation where the diffusion coefficient is stochastic requires a nontrivial use of stochastic analysis.

For stochastic linear distributed parameter equations, with multiplicative Brownian motion, Da Prato and Zabczyk [2] have given explicit solutions. The approach used here for stochastic linear distributed parameter systems with multiplicative fractional Brownian motion is motivated by [2], but the analysis for this case requires additional methods as contrasted with [2].

A standard fractional Brownian motion \((β^H(t), t ≥ 0)\) with Hurst parameter \(H ∈ (0, 1)\) is a Gaussian process with continuous sample paths such that 

\[ E[β^H(t)] = 0 \]

\[ E[β^H(s)β^H(t)] = \frac{1}{2} [s^{2H} + t^{2H} - |s - t|^{2H}] \]

for \(s, t ∈ \mathbb{R}_+\). It is clear that for \(H = 1/2\), the process is a (standard) Brownian motion. In this paper, it is assumed that \(H ∈ (1/2, 1)\). These processes have a long range dependence (e.g., [3]) in addition to a self similarity. The Hurst parameter \(H ∈ (1/2, 1)\) has been estimated from empirical data from many applications.

The stochastic equation considered here is given by

\[ dX(t) = A(t)X(t)dt + \sum_{j=1}^{k} B_j X(t)dB^H_j(t) \]

\[ X(0) = x_0 ∈ V, \]

where \(t ∈ [0, T]\), \(X(t) ∈ V, V\) is a separable Hilbert space, \((β^H_j(t), t ≥ 0, j = 1, \ldots, k)\) is a family of independent, standard fractional Brownian motions with a fixed Hurst parameter \(H ∈ (1/2, 1)\) defined on a complete probability space \((Ω, F, P)\), \((A(t), t ∈ [0, T])\) and \((B_1, \ldots, B_k)\) are typically linear, densely defined operators on \(V\).

The following assumptions are used subsequently.

\( (A1) \) The family of closed operators \((A(t), t ∈ [0, T])\) defined on a common domain \(D := \text{Dom}(A(t))\) for \(t ∈ [0, T]\) generates a strongly continuous evolution system \((U_0(t, s), 0 ≤ s ≤ t ≤ T)\).

\( (A2) \) The linear operators \(B_1, \ldots, B_k\) generate mutually commuting, strongly continuous groups denoted \(S_1, \ldots, S_k\), respectively, which commute with \(A(t)\) for each \(t ∈ [0, T]\) on \(D\). For \(j, m ∈ \{1, \ldots, k\}\), \(\text{Dom}(B_jB_m) ⊃ D\), \(\text{Dom}(A^* (t)) = D^*\) for each \(t ∈ [0, T]\), and \(D^* ⊃ \bigcap_{j,m=1}^{k} \text{Dom}(B_j^*B_m^*)\), where \(^*\) denotes the adjoint.

\( (A3) \) The family of linear operators \(A(t) := A(t) - Ht^{2H-2} × \sum_{j=1}^{k} B_j^2\) generates a strongly continuous evolution system on \(V\) and \(\text{Dom}(A(t)) = D\) for each \(t ∈ [0, T]\).

Additionally some more specific conditions on \(A(t)\) and \((B_1, \ldots, B_k)\) are made that ensure \((A1)\) and \((A3)\). These conditions are useful in applications to stochastic partial differential equations (SPDEs) of parabolic type.

\( (H1) \) For each \(t ∈ [0, T]\), the linear operator \(A(t)\) is a closed densely defined operator in \(V\) whose resolvent set \(\rho(A(t))\) contains the half-plane \(\text{Re} λ ≥ ω_0\) for some fixed \(ω_0 ∈ \mathbb{R}\) and

\[ |R(λ, A(t))|_{L(V)} ≤ \frac{M}{1 + |λ + ω_0|}, \quad \text{Re } λ ≥ ω_0 \]

for some real number \(M\) that does not depend on \(t ∈ [0, T]\).

The condition \((H1)\) implies that \(A(t)\) generates an analytic semigroup [6] for each \(t ∈ [0, T]\) so it can be assumed by a translation that \(ω_0 = 0\).

\( (H2) \) For each \(t ∈ [0, T]\), \(\text{Dom}(A(t)) = D = \text{Dom}(A)\) where \(A := -A(0)\) and \(A(t)A^{-1}\) is a Hölder continuous function in \(L(V)\), or, equivalently, the following inequality is satisfied

\[ |A(t) - A(s)|_{L(D,V)} ≤ K|t - s|^γ \]

for \(s, t ∈ [0, T], K ∈ \mathbb{R}_+, \text{ and } γ ∈ (0, 1]\).

It is known (e.g., [7], Theorem 5.2.1) that the assumptions \((H1)\) and \((H2)\) imply \((A1)\) and, furthermore, \(\text{Range}(U_0(t, s)) ⊂ D\)

\[ \left| \frac{∂}{∂t} U_0(t, s) \right|_{L(V)} = |A(t)U_0(t, s)|_{L(V)} ≤ \frac{c}{t - s}, \quad 0 ≤ s < t ≤ T \]

\[ |U_0(t, s)|_{L(D)} ≤ c. \]

The following proposition verifies that \((H1)\) and \((H2)\) imply \((A3)\) under some conditions on \((B_j^*, j = 1, \ldots, k)\).

**Proposition 1** Let \((B_j^2, j = 1, \ldots, k)\) be a family of closed operators such that \(\text{Dom}(B_j^2) ⊃ \text{Dom}(A^α)\) for some \(α ∈ (0, 1)\). If \((H1)\) and \((H2)\) are satisfied, then the family of operators \((A(t), t ∈ [0, T])\) with \(\text{Dom}(A(t)) = D\) generates a strongly continuous evolution system on \(V\), that is, \((A3)\) is satisfied.
Definition 1 A $\mathbb{B}([0,T]) \otimes \mathcal{F}$ $V$-valued stochastic process $(X(t), t \in [0,T])$ is said to be 
(i) a strongly solution of (1) if $X(t) \in D$ a.s. $\mathbb{P}$ and 
$$X(t) = x_0 + \int_0^t A(s)X(s)ds + \sum_{j=1}^k \int_0^t B_jX(s)d\beta^H_j(s) \quad \text{a.s. } \mathbb{P}$$ 
for $t \in [0,T]$ 
(ii) a weak solution of (1) if for each $z \in D^*$ 
$$\langle X(t), z \rangle = \langle x_0, z \rangle + \int_0^t \langle X(s), A^*(s)z \rangle ds + \sum_{j=1}^k \int_0^t \langle X(s), B^*_jz \rangle d\beta^H_j(s) \quad \text{a.s. } \mathbb{P}$$ 
for $t \in [0,T]$ 
(iii) a mild solution of (1) if 
$$X(t) = U_0(t,s)x_0 + \sum_{j=1}^k \int_0^t U_0(t,s)B_jX(s)d\beta^H_j(s) \quad \text{a.s. } \mathbb{P}$$ 
for $t \in [0,T]$.

The definition of the stochastic integrals in (2,3,4) can be found in [1], [4], [5].

The following theorem is the main result in the paper. It demonstrates that (A1)–(A3) are sufficient for weak or strong solutions. Furthermore, the solution is given explicitly in terms of the fractional Brownian motions and the linear operators that characterize the linear equation (1).

**Theorem 1** In (A1)–(A3) are satisfied, then there is a weak solution of (1). Moreover, if $x_0 \in D$, then there is a strong solution. If $B_j \in \mathcal{L}(V)$, $j = 1, \ldots, k$, then there is a mild solution. In each of these cases, the solution is given explicitly as 
$$X(t) = \prod_{j=1}^k S_j(\beta^H_j(t))U(t,0)x_0$$ 
for $t \in [0,T]$, where $U(\cdot, \cdot)$ is the evolution system for $A$.

The following examples show that noise may have a stabilizing or destabilizing effect.

2. EXAMPLES

**Example 1 (Noise that has a stabilizing effect)** Let us consider the Cauchy problem 
$$\frac{\partial u}{\partial t} = \sum_{i,j} a_{ij}(t, \xi) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \sum_{i} d_i(t) \frac{\partial u}{\partial \xi_i}$$ 

Assume H"older continuous coefficients and uniform ellipticity. 
$$X(t) = S_1(t)U(t,0)x_0$$ 
$S_1$ is the shift operator 
$$S_1(t)(\xi) = (\xi_1 + b_1t, \ldots, \xi_d + b_d t)$$ 
$U(t,0)$ is the solution to 
$$\frac{\partial u}{\partial t} = \sum_{i,j} \tilde{a}_{ij}(t) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \sum_{i} d_i(t) \frac{\partial u}{\partial \xi_i} + cu$$ 
$$\tilde{a}_{ij}(t) = a_{ij}(t) - Ht^{2H-1}b_ib_j.$$ 
Stability is reduced to deterministic problem.

**Example 2 (Noise that has a destabilizing effect)** 
$$\frac{\partial u_1}{\partial t} = \Delta u_1 + b_1u_2\beta^H(t)$$ 
$$\frac{\partial u_2}{\partial t} = \Delta u_2 + b_1u_1\beta^H(t)$$ 
$$V = (L^2(\theta))^2$$ 
$$B = \begin{pmatrix} 0 & b_1I \\ b_2I & 0 \end{pmatrix}$$ 
$$B^2 = b_1b_2I.$$ 
Assume $b_1b_2 < 0$. $B$ is a negative operator 
$$X(t) = \exp\left[-\frac{1}{2}b_1b_2t^{2H}\right]S_B(\beta^H(t))S_A(t)x_0$$ 
$S_B$ is the group generated by $B$ (periodic in $t$) 
$$A = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}.$$ 
Let $x_0 = \epsilon_{\alpha_0}$ be an eigenvector for the first eigenvalue $\alpha_0 > 0$ 
$$\lim_{t \to \infty} X(t) = \infty \quad \text{a.s.}$$

3. APPLICATIONS OF FRACTIONAL BROWNIAN MOTION CALCULUS TO EPILEPSY

Epilepsy is a condition where a person has unprovoked seizures at two or more separate times in his/her life. A seizure is an abnormal electrical discharge within the brain resulting in involuntary changes in movement, sensation, perception, behavior, and/or level of consciousness. It is estimated that 1% of the populations of industrialized countries have epilepsy whereas 5–10 have epilepsy. In the United States this number is
significantly more than the people who have Parkinson’s disease, muscular dystrophy, multiple sclerosis, AIDS (plus the estimated (HIV), or Alzheimer’s disease.

The University of Kansas Medical Center Comprehensive Epilepsy Center in conjunction with Flint Hills Scientific, LLC has the largest recorded collection of long term patient seizure data. Flint Hills Scientific (FHS) has used this data to develop real time seizure prediction algorithms. These algorithms outperform any other prediction algorithm. Recently FHS has initiated electrical stimulation control with a seizure prediction algorithm to prevent the occurrence of seizures.

There is some published evidence that the seizure periods of brain waves of some patients have a long range dependence with nonseizure periods. Based on some initial work, it seems that the estimates of the Hurst parameter, which is a characterization of the long range dependence in fractional Brownian motion, have a noticeable change prior and during a seizure. An extensive analysis of the seizure data that is described above to estimate the Hurst parameter has been initiated.

To understand the seizure phenomenon better, the electrical brain signals has been modelled using a fractional Brownian motion (FBM) that is obtained from an estimate of the Hurst parameter, as well as some processes that are obtained from a FBM, especially linear stochastic differential equations with FBM. Some parameter identification algorithms for linear stochastic differential equations that have been developed [10] from a pseudo least squares approach are used. The identification algorithms use the recently developed stochastic calculus for fractional Brownian motion.

4. FOCUS ON EDUCATION

Let me quote my former student’s recommendation given to current students who are considering careers in industry:

“Always be alert in your analysis of what you are doing—ask yourselves why things are the way they are—see if you can explain what you observe. Make sure that you consider how even the most abstract mathematics you do can be applied to real life problems.” —Mark Frei, technical director of Flint Hills Scientific, LLC

5. FUTURE WORK

Research will focus on the balance between statistical efficiency and computational complexity in the presence of large data set.

REFERENCES


