Ch 6.1 Area between two curves
* vertically simple (you have a clear top / bottom function)
  then you integrate w.r.t \( x \)
  \[ \int_{x_1}^{x_2} (y_{\text{top}} - y_{\text{bottom}}) \, dx \]
  * the bounds should match the variable you're taking

* horizontally simple (you have a clear left / right function)
  then you integrate w.r.t \( y \)
  \[ \int_{y_1}^{y_2} (x_{\text{right}} - x_{\text{left}}) \, dy \]
  written in terms of \( y \)

Sometimes the regions are not simple as a whole so you need to break them up into different integrals

Example 1) Find the area between

\[ y = 2x + 3 \quad y = 13 - x^2 \quad x = -1 \quad x = 2 \]

Vert. Simple:
\[ \int_{-1}^{2} (g(x) - f(x)) \, dx \]

\[ = \int_{-1}^{2} (13 - x^2) - (2x + 3) \, dx \]

\[ = \int_{-1}^{2} -x^2 - 2x + 10 \, dx \]

(integrate normally)
Example 2) \( y = \sqrt{x} \) \( y = \frac{1}{2} x \)

\( x = 25 \)

Vert simple. But at \( x = 4 \)
Top: Bottom from switch so

\[
\int_4^9 -\frac{1}{2} x \, dx + \int_{25}^4 \frac{1}{2} x - \sqrt{x} \, dx
\]

Then integrate individually.

Example 3) \( y = 13x, y = x^2 - 4 \)

Intersect at

\[ x^2 - 4 = 3x = 0 \]
\[ x^2 - 3x - 4 = 0 \quad (x-4)(x+1)=0 \]
\[ x=4 \quad \checkmark \quad \text{if} \quad x=1 \quad \text{then} \quad |3|=3 \neq (1)^2-4=-3 \]
\[ x^2 - 4 = -3x \Rightarrow x^2 + 3x - 4 = 0 \quad (x+4)(x-1)=0 \]
\[ x=-4 \quad \checkmark \quad \text{if} \quad x=1 \quad \text{then} \quad |3|=3 \neq (1)^2-4=-3 \]

Vert simple

\[
\int_4^9 13x - (x^2-4) \, dx
\]

But we can't integrate \( 1 \times 1 \) thus we have to break it down

By using definition of absolute value

For \( x < 0 \) \( 13x = -3x \)
For \( x > 0 \) \( 13x = 3x \)

\[
\int_{-4}^{0} -3x - (x^2-4) \, dx + \int_{0}^{4} 3x - (x^2-4) \, dx
\]

Then integrate individually.
Example 4) Find a < x < b which divides the region bounded by \( y = 4x^2 \) and \( y = 16 \) into two equal areas.

\[
\text{Area 1:} \quad \int_0^b 2\sqrt{y} \, dy = \int_b^{16} 2\sqrt{y} \, dy
\]

\[
\sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}
\]

\[
\frac{4}{3} y^{3/2} \bigg|_b^{16} = \frac{4}{3} y^{3/2} \bigg|_0^b = (64 - b^{3/2}) = 3^{2/3} = b
\]

6.2 Volumes

PART I: Volumes with Cross Sections

- Cylinder: \( V = Ah \)
- Circular Cylinder: \( V = \pi r^2 h \)
- Rectangular Box: \( V = l \cdot w \cdot h \)

(Dr. Brennan taught the Riemann sum of volumes in terms of right cylinders and it came down to setting up an integral of an area function)
Example 1) Find the volume of the solid's:

The base of S is a circular disk with radius 4r. Parallel cross sections perpendicular to the base are squares.

Let \( R = 4r \)

then \( A(x) = (2y)^2 = 4(R^2 - x^2) \)

\( y = \sqrt{R^2 - x^2} \) thus

\[ V = \int_{-R}^{R} A(x) \, dx \]

\[ = \int_{-R}^{R} 4(R^2 - x^2) \, dx \]

\[ = 4\left( R^2x - \frac{1}{3}x^3 \right) \Big|_{x=-R}^{x=R} \]

\[ = \frac{16}{3}R^3 = \frac{16}{3} \left( \frac{2}{3} \right)r^3 \]

*depending on \( x \), the area of your square will change so want to find the area of the square in terms of the changing \( x \)

(Note: The same exact set up will work for \( \int A(y) \, dy \))

Example 2) Find the volume of the solid S:

The base of S is the triangular region with vertices (0,0), (4,0), (0,4) with cross sections perpendicular to the y-axis are equilateral triangles.

Area of equilateral triangle

\[ \frac{\sqrt{3}}{4} a^2 \]

Length of side

\[ y = 4 - x \]

\[ x: 0 \rightarrow 4 \]

\[ y: 0 \rightarrow 4 \]

Area of corresponding \( x \) is

\[ A(y) = \frac{\sqrt{3}}{4} x^2 = \frac{\sqrt{3}}{4} (4-y)^2 \]

\[ V = \int_{0}^{4} A(y) \, dy = \frac{\sqrt{3}}{4} \int_{0}^{4} (4-y)^2 \, dy \]

\[ = \frac{\sqrt{3}}{4} \left[ 16y - 4y^2 + \frac{1}{3}y^3 \right]_{0}^{4} \]

\[ = \frac{16\sqrt{3}}{3} \]
**PART 2: Volumes with Washers**

*Solid of Revolution* is a 3-d figure obtained from rotating a 2-d region about x-axis.

* Cross sections perpendicular to x-axis are circular.
* Rotating a line segment perpendicular to the x-axis makes a washer.

\[ \text{Area } A = \pi \left( \text{out}^2 - \text{in}^2 \right) \]

**Example 1**) Consider the solid obtained by rotating the region bounded by
\[ y = \sqrt{x}, \quad y = x \]
about the line \( x = 6 \).

Find the volume.

Inner R: \( 6 - x \geq y \)
Cutter R: \( 6 - \sqrt{x} \)

\[ V = \int_{0}^{1} \pi \left( (6 - y^2)^2 - (6 - y^2)^2 \right) \, dy \]
\[ = \int_{0}^{1} \pi \left( y^4 - 12y^2 + 12y \right) \, dy \]
\[ = \frac{28}{15} \pi \]
Example 2) Consider the solid obtained by rotating the region
bounded by \( y = \frac{x^2}{25} \) \( x=2 \) \( y=0 \) about the y-axis.

Find the volume.

So \( \pi \int_0^{\frac{4}{25}} (2^2 - (5\sqrt{y})^2) \, dy = \pi \int_0^{\frac{4}{25}} 4 - 25y \, dy = \pi \left( \frac{4y - \frac{25}{2}y^2}{y} \right)_0^{\frac{4}{25}} = \frac{8\pi}{25} \)
Some problems of finding volumes are extremely difficult to handle using cross-section or washer methods. In these cases we use shell method.

Consider the cylindrical shell

\[ h \]

(See Book pg 449)

\[ r_2 - \text{outer radius} \]

\[ r_1 - \text{inner radius} \]

\[ h - \text{height} \]

\[ \Delta r - \text{thickness} \hspace{1cm} (\Delta r = r_2 - r_1) \]

Volume is calculated by subtracting the volume of the inside cylinder \( V_1 \) with radius \( r_1 \) from the volume of the outer cylinder \( V_2 \) with radius \( r_2 \).

\[ V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h = \pi (r_2^2 - r_1^2) h = \pi (r_2 + r_1)(r_2 - r_1) h = \pi (r_2 + r_1) h \Delta r \]

If we let \( r \) be the average radius of the shell, \( r = \frac{r_2 + r_1}{2} \) then

\[ V = 2\pi \cdot r \cdot h \cdot \Delta r \]

\[ \frac{\text{circumference}}{\text{height}} \cdot \text{thickness} \]

So, what the shell method does is uses these cylinders to calculate the volume of a figure (see pg 450). We set it up by

\[ \int_a^b \frac{2\pi x}{\text{circumference}} \cdot f(x) \cdot \frac{\text{height}}{\text{thickness}} \, dx \]
Solid of Revolution: solids obtained by revolving a region about a line.

In general, we can calculate the volume of a solid of revolution by

\[ V = \int_{a}^{b} A(x) \, dx \quad \text{or} \quad V = \int_{c}^{d} A(y) \, dy \]

1. If the cross section is a disc or semi-circle
   
   We find the radius of disc/semi-circle;
   
   \[ A = \pi r^2 \quad \text{or} \quad A = \pi r^2 / 2 \]
   where \( r \) is in terms of \( x \) or \( y \) depending on the simplicity of the graph.

2. If the cross section is a triangle
   
   We find the base height; base of the triangle;
   
   \[ A = \frac{1}{2} b \cdot h \]

3. If the cross section is a square
   
   We find the length of one of the sides;
   
   \[ A = l^2 \]

4. If there is a hole in your figure, you will
   
   use either the washer or shell method.