Parametric Curves (1.7)

Parametric equations are of the form 
\[ x = f(t) \quad y = g(t) \]

where \( t \) is called the parameter (a third variable that both \( x \) and \( y \) depend on).

As \( t \) changes, the point \((x, y) = (f(t), g(t))\) changes. A curve traced out by this \((x, y)\) is called a parametric curve.

What you should be able to do with parametric curves:
(1) Sketch curve by eliminating the \( t \)-parameter.

Example:
\[ x = \sin(t) \quad y = \cos(t) \quad 0 < t < \pi/2 \]

\[ \frac{y}{x} = \frac{\cos(t)}{\sin(t)} = \frac{1}{x} \]

For \( 0 < t < \pi/2 \) we have \( 0 < x < 1 \) \( y > 1 \) thus the graph is \( y = \sqrt{x} \) with \( y > 1 \)

(2) Find the derivative given a parametric equation.

Example: given 
\[ x = 1 - e^{-t} \quad y = e^{\sqrt{t}} \quad 0 \leq t \leq 2 \]

\[ \frac{dy}{dx} \text{ in terms of } t \text{ is: } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \]

\[ \frac{dy}{dt} = \frac{1}{2} e^{\sqrt{t}} \quad \frac{dx}{dt} = -e^{-t} \quad \frac{dy}{dt} = \frac{e^{\sqrt{t}}}{2} \]

Equation of the tangent line to the curve at \( t = 1 \):

(i) Find point:
\[ x = 1 - e^{-1} = 1 - \frac{1}{e} \quad y = e^{1/2} \]

(ii) Find slope:
\[ \frac{dy}{dx} \bigg|_{t=1} = e^{3/2} / 2 \]

(iii) Write equation:
\[ y - e^{1/2} = \frac{e^{3/2}}{2} \left( x - (1 - \frac{1}{e}) \right) \]
(3) Calculate area under curve enclosed by a parametric equation:

*Example: \( x = \cos^3(t) \), \( y = \sin^2(t) \) \( 0 \leq t \leq \pi \)

*Know: area under \( x = f(t) \), \( y = g(t) \) for \( a \leq t \leq b \) with

\[
\alpha = f(a) \quad \beta = f(b)
\]

\[
\int_{\alpha}^{\beta} y \, dx = \int_{\alpha}^{\beta} g(t) f'(t) \, dt
\]

So we have:

\[
x = f(t) = \cos^3(t) \quad y = g(t) = \sin^2(t) \quad 0 \leq t \leq \pi \quad (a = 0, b = \pi)
\]

\[
\alpha = f(0) = 1 \quad \beta = f(\pi) = 0
\]

\[
\int_{0}^{\pi} y \, dx = \int_{0}^{\pi} \sin^2(t) \cdot 3\cos^2(t) \cdot -\sin(t) \, dt
\]

\[
= -3 \int_{0}^{\pi} \sin^3(t) \cos^2(t) \, dt
\]

\[
= -3 \int_{0}^{\pi} \left( \cos^3(t) \cdot 3\cos^2(t) \cdot 0 \cdot (\sin(t)) \right) \, dt
\]

\[
= -3 \int_{0}^{\pi} [\cos^2(t) - \cos^4(t)] (\sin(t)) \, dt
\]

\[
u = \cos(t) \quad du = -\sin(t) \, dt
\]

\[
t=0 \Rightarrow u=1 \quad t=\pi \Rightarrow u=-1
\]

\[
= 3 \int_{1}^{-1} u^2 - u^4 \, du
\]

\[
= 3 \left[ \frac{1}{3} u^3 - \frac{1}{5} u^5 \right]_{1}^{-1}
\]

\[
= 3 \left( \frac{1}{3} + \frac{1}{5} \right) - \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{-4}{5}
\]
Applications of Integration (6.4-6.7)

(A) Arc length: If a smooth curve with parametric equations \( x = f(t) \), \( y = g(t) \) \( a \leq t \leq b \) is traversed exactly once as \( t : a \to b \) then
\[
L = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt
\]

If we have \( f(x) = y \) on \( a \leq x \leq b \) then
\[
L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx
\]

If we have \( g(y) = x \) on \( c \leq y \leq d \) then
\[
L = \int_{c}^{d} \sqrt{1 + (g'(y))^2} \, dy
\]

*Smooth means the derivatives \( f'(t) \), \( g'(t) \) are continuous. Not simultaneously 0 on \((a,b)\).

Special application of arc length
Surface area of the solid of revolution which results from rotating \( y = f(x) \) on the interval \([a,b]\) about the horizontal line \( y = c \) is
\[
\int_{a}^{b} 2\pi (\text{radius}) \cdot (\text{segment length}) = \int_{a}^{b} 2\pi |c - f(x)| \sqrt{1 + (f'(x))^2} \, dx
\]

(B) Average value: Given \( f(x) \) on \([a,b]\), the average value of \( f(x) \) on \([a,b]\) is
\[
\overline{f}_{[a,b]} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\]

*MVT for Integrals*: If \( f \) is continuous on \([a,b]\) then there exists a \( c \in [a,b] \) such that
\[
\int_{a}^{b} f(x) \, dx = f(c)(b-a)
\]
(3) Applications in Science:

Formulas to know:
- \( W = F \cdot d \) (work = force \cdot distance)
- \( F = m \cdot g \) (force = mass \cdot acceleration)
- gravity

- Work done in moving an object from \( a \) to \( b \)
  \[ W = \int_{a}^{b} f(x) \, dx \]
  where \( f(x) \) is a force acting on the object

- Hooke's Law: the force required to maintain a spring stretched \( x \) units beyond its natural length \( d \)
  proportional to \( x \): \( f(x) = kx \)
  \( k > 0 \) is the spring constant
  \( x \) is not too large

- Mass/Center of mass with uniform density \( \rho \):
  Region between 2 curves \( y = f(x) \) and \( y = g(x) \)
  on \([a, b]\) \( f(x) \geq g(x) \) on \([a, b]\) with density \( \rho \)
  the mass is
  \[ m = \rho \int_{a}^{b} (f(x) - g(x)) \, dx \]

  the center of mass \((\bar{x}, \bar{y})\) is
  \[ \bar{x} = \frac{1}{m} \rho \int_{a}^{b} x(f(x) - g(x)) \, dx \]
  \[ \bar{y} = \frac{1}{m} \rho \int_{a}^{b} \frac{1}{2} (f(x)^2 - g(x)^2) \, dx \]
Examples of parametric equations:

1) Find the exact length of the curve:
   \[ x(t) = (t+2t^2), \quad y(t) = 3+3t^3, \quad 0 \leq t \leq 4 \]
   \[ x'(t) = 24t, \quad y'(t) = 24t^2 \]

   \[ \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{576t^4 + 576t^4} = 24t \sqrt{1 + t^2} \]

   \[ L = \int_0^4 24t \sqrt{1 + t^2} \, dt = 24 \left[ u^{3/2} \right]_1^4 = 8 \sqrt{17} - 1 \]

2) Find the surface area of a spherical cap of height \( h = 5 \text{ m} \),
   radius \( R = 7 \text{ m} \):

   \[ S_A = 2\pi Rh \]
   \[ = 70\pi \]
   \[ \approx 219.9 \text{ m}^2 \]

3) Find the max arc length of \( y = f(x) \) over \( [1, 9] \) if \( 0 < f'(x) < 1 \)

   \[ 0 < f'(x)^2 < 1 \Rightarrow 1 < 1 + f'(x)^2 < 2 \]

   \[ 1 < \sqrt{1 + f'(x)^2} < \sqrt{2} \]

   \[ \Rightarrow \int_a^b 1 \, dx = (b-a) < \int_a^b \sqrt{1 + f'(x)^2} \, dx = L < \int_a^b \sqrt{2} \, dx = (b-a)\sqrt{2} \]

   But \( a = 1, b = 9 \text{ so} \)

   max arc length is \( (9-1)\sqrt{2} = 8\sqrt{2} = 11.31 \)
Examples of Applications of Integration:

1. Find the number $b$ such that the average value of $f(x) = 7 + 10x - 6x^2$ on $[0, b]$ is 8.

$$8 = \frac{1}{b} \int_0^b (7 + 10x - 6x^2) \, dx$$

$$8 = \frac{1}{b} \left[ 7b + 5b^2 - 2b^3 \right] = 7 + 5b - 2b^2$$

$$-2b^2 + 5b - 1 = 0 \quad \text{using quadratic formula}$$

$$b = \frac{5 \pm \sqrt{17}}{4} \quad \text{(Both are positive so both values are answers)}$$

2. A rope 70 ft long weighs 51 lbs and hangs over the edge of a building 120 ft high. How much work is done in pulling the rope to the top?

(a) The rope weighs $\frac{5}{70} = 0.0714 \text{ lb/ft}$

Total Work

$$W = \int_0^{120} 0.0714 \cdot x \, dx = 0.0357 \cdot x^2 \Bigg|_0^{120} = 175$$

(b) When half the rope is pulled to the top of the building, the top half is

$$W_1 = \int_0^{35} 0.0714 \cdot x \, dx = 0.0357 \cdot x^2 \Bigg|_0^{35} = 43.75$$

$$(W - W_1 = W_2)$$

The bottom half is lifted 25 feet with work

$$W_2 = \int_0^{70} 0.0714 \cdot x \, dx = 0.0357 \cdot x^2 \Bigg|_0^{70} = 131.25$$
13) If 0.6 J of work is needed to stretch a spring from 9 cm to 11 cm and 1 J is needed to stretch from 11 cm to 13 cm, what is the natural length of the spring?

Let $L$ be natural length. Then

$$0.6 = \int_{0.09}^{0.11} -L \cdot k \cdot dx = \frac{1}{2} k \left[ (0.11 - L)^2 - (0.09 - L)^2 \right] \quad (*)$$

$$1 = \int_{0.11}^{0.13} -L \cdot k \cdot dx = \frac{1}{2} k \left[ (0.13 - L)^2 - (0.11 - L)^2 \right] \quad (***)$$

Simplifying gives:

$(*)$ \quad $1.2 = k \left( 0.0040 - 0.044L \right)$

$(***)$ \quad $2 = k \left( 0.0048 - 0.044L \right)$

Subtracting these equations gives:

$0.8 = 0.0008k \Rightarrow k = 1000$

Plugging this into either equation we get $L = 7$ cm.

(Hooke's Law: force to hold spring x m beyond its natural length $\Rightarrow f(x) = kx$) work done (in J) is the integral of force.

14) A tank is full of water. Find the work required to pump out water from the spout (g = 9.8)

[Diagram of a rectangular prism with dimensions]

A rectangular slice of water $\Delta x$ m thick, lying $x$ m above the bottom has width $x$; volume $15 \times \Delta x$ m$^3$.

It weighs $9.8 \times 1000 \times 15 \times \Delta x$ N; must be lifted $(7-x)$ m.

So work need is about $(9.8 \times 1000) (7-x)(15 \times \Delta x)$ J; thus the total work is

$$N = \int_0^4 (9.8 \times 1000) (7-x) 15 \times \Delta x = (9.8 \times 10^3) \left[ \frac{155 x^2 - 9 x^3}{2} \right]_0^4 = (9.8 \times 10^3) \left[ 5094 \right] = 5.096 \times 10^6 J$$
(5) A tank is full of water. Find the work required to pump the water out of the spout.

Use: weight of water $62.5 \text{ lb/ft}^2$; $\pi = 3.14$

Let $x$ be depth (in ft) below the spout at the top of the tank. A horizontal disc-shaped 'slice' of 15ft of water $\Delta x$ thick & lying at coordinate $x$ has radius

$$\frac{4}{15} (30-x)$$

By similar triangles we get

$$\frac{d}{15} = \frac{4}{15}$$

so

$$r = 4 + d = 4 + \frac{4}{15} (15-x) = \frac{4}{15} (30-x)$$

Volume $\pi r^2 \Delta x = \pi \left( \frac{4}{15} (30-x) \right)^2 \Delta x$ It weighs about

$$62.5 \pi \left( \frac{16}{225} \right) (30-x)^2 \Delta x$$

It must be lifted $x$ ft so

the total work needed is:

$$w = \int_0^{15} \frac{62.5 \pi \cdot 16}{225} (30-x)^2 \cdot x \, dx = \frac{62.5 \cdot 16 \pi}{225} \int_0^{15} 900 - 60x + x^2 \, dx \approx 6.48 \times 10^5$$

(6) The # of mosquitoes is increasing at a rate of

$$n(t) = 1700 + 100e^{0.7t}$$

It measured in weeks. By how much does the mosquito population increase between 5th & 9th week?

$$n(9) - n(5) = \int_5^9 1700 + 100e^{0.7t} \, dt = 1700 + 100 \cdot \frac{e^{0.7}}{0.7} \left|_5^9 \right.$$

$\approx 14106.521$