RAINBOW COLORINGS OF SOME GEOMETRICALLY DEFINED UNIFORM HYPERGRAPHS IN THE PLANE

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1 Introduction

A hypergraph is a pair (V,E) in which V is a non-empty set, the set of vertices of the hypergraph, and E is a collection of non-empty subsets of V, called hyperedges, or for short, edges. If V is colored (partitioned) an edge e ∈ E is rainbow, with respect to the coloring, if and only if no two distinct vertices on e have the same color – i.e., different vertices on e belong to different parts of the partition. A coloring of V is rainbow (for the hypergraph (V,E)) if and only if every e ∈ E is rainbow with respect to the coloring. The rainbow chromatic number of a hypergraph G=(V,E), denoted \( \chi_r(G) \), is the smallest cardinal number of colors in a rainbow coloring of G.

A hypergraph G = (V,E) is k-uniform if and only if \(|e| = k\) for all e ∈ E; here \(|e|\) stands for the cardinality of e. A 2-uniform hypergraph is a simple graph or graph, for short. A rainbow coloring of a graph G is an ordinary proper coloring of G.

In this note the hypergraphs will be the uniform hypergraphs (R², E(F)) in which R² stands for the Euclidean plane, \( \emptyset \neq F \subseteq R^2 \) is finite, and E(F) is the collection of all copies of F in R². “Copy” can mean two different things: in one definition, a copy of F is the image of the original F under a mapping consisting

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of a translation followed by a rotation, and in the other definition
you are allowed translation, rotation, and reflection in a line. It
will turn out that the two possibly different hypergraphs \((R^2, E(F))\)
arising from these different definitions have the same rainbow
chromatic number.

For \(u,v \in R^2\), let \(|u-v|\) denote the usual Euclidean distance
between \(u\) and \(v\). For \(D \subseteq (0,\infty)\), the distance graph on \(R^2\) defined
by \(D\), denoted \(G(R^2, D)\), is the graph with vertex set \(R^2\), with \(u,v \in R^2\) adjacent in the graph if and only if \(|u-v| \in D\). Let \(\chi(R^2, D)\)
denote the chromatic number of this graph—which is also the
rainbow chromatic number of the graph, because each edge has
two elements. When \(|D| = 1\), \(\chi(R^2, D)\) is the famous chromatic
number of the plane, often denoted \(\chi(R^2,1)\), known to be one of
4,5,6,7 [2].

For each positive integer \(k\), the \(k^{th}\) Babai number of \(R^2\) is:
\[B_k(R^2) = \max[\chi(R^2, D); D \subseteq (0,\infty)\ and \ |D| = k]\]

Clearly \(B_1(R^2) = \chi(R^2,1)\). It is shown in [1] that \(B_{k+t}(R^2) \leq B_k(R^2)B_t(R^2)\) for all positive integers \(k,t\); therefore \(B_k(R^2) \leq B_1(R^2)^k = \chi(R^2,1)^k \leq 7^k\).

**Proposition:** Suppose that \(F \subseteq R^2\) and \(2 \leq |F| \leq \infty\). Let \(D(F)\) be the
set of distances realized between points of \(F\). A coloring of \(R^2\) is a
rainbow coloring of \((R^2, E(F))\) if and only if it is a proper coloring
of \(G(R^2, D(F))\).

Proof: If a coloring of \(R^2\) “forbids” every distance in \(D(F)\), then no
two points in any copy of \(F\) can have the same color, so the
coloring is a rainbow coloring of \((R^2, E(F))\). On the other hand,
suppose we have a rainbow coloring of \((R^2,E(F))\); suppose \(u,v \in R^2, |u-v| \in D(F)\). Clearly \(u\) and \(v\) are together in some copy of \(F\),
so \(u\) and \(v\) must be colored differently. Thus the coloring is a
proper coloring of \(G(R^2, D(F))\). ■
**Corollary 1:** With $F$ and $D(F)$ as in the Proposition, $\chi_r(R^2,E(F)) = \chi(R^2,D(F)) \leq B_{|D(F)|}(R^2) \leq B_{|F|}(R^2) \leq 7^{\binom{|F|}{2}}$.

So, for instance, if $|F|=3$, then $\chi_r(R^2,E(F)) \leq 7^3 = 343$. In what follows, we will see much lower upper estimates than 343 for $\chi_r(R^2,E(F))$ for sets $F \subseteq R^2$, $3 \leq |F| < \infty$, satisfying certain conditions.

The main trick: give a coloring of $R^2$ and then figure out for which $F \subseteq R^2$ the coloring is a rainbow coloring of $(R^2,E(F))$.

For $F \subseteq R^2$, $\infty > |F| \geq 2$, let

$$M(F) = \max[|u-v|; u,v \in F] \text{ and } m(F) = \min[|u-v|; u,v \in F u \neq v]$$

2 Results

**Theorem 1:** To forbid any set of distances contained in an interval $[a, a\sqrt{7}/2]$, a $> 0$ requires at most 7 colors.

**Proof:** Consider a regular hexagon of diameter $a$. This hexagon will have sides of length $a/2$ and height $\sqrt{3}a/2$. Color all the points inside this hexagon a color, $c$. Color the bottom three sides and the bottom two corners $c$ as well. If the plane is tiled with these hexagons, and each hexagon tile is colored as described with some color, then each point of the plane will be colored once.

Then construct six other hexagons of different six colors but same dimensions. Put these hexagons around the first hexagon forming a Hadwiger tile (Figure 1).

Stack these Hadwiger tiles such that the bottom hexagon of the upper Hadwiger tile is adjacent to the top and upper right hexagons of the bottom Hadwiger tile. This can be done indefinitely, covering an entire strip of the plane (Figure 2).
Figure 1: A Hadwiger tile.

Figure 2: A stack of Hadwiger tiles. This can be done indefinitely.

Then make identical stacks and place them to the left and right of the initial stack (Figure 3). This also can be done indefinitely. This will cover the whole plane.

By symmetry, each hexagon will be the same distance away from any nearest hexagon of the same color. So we will calculate the distances from the center black hexagon to the surrounding hexagons. It is apparent that the closest hexagon of the same color is in one of the adjacent Hadwiger tiles.
**Figure 3:** Placing the stacks of Hadwiger tiles next to each other, the plane can be covered.

**Figure 4:** The lines show the shortest distance from the black hexagon in the center Hadwiger tile to the other black hexagons in the surrounding Hadwiger tiles.
The shortest distance between the center black hexagon and the top right black hexagon is \( \sqrt{7}a/2 \). Therefore, it is impossible to have two points be the same color if they are in the range \([a, \sqrt{7}a/2]\). Since we have colored the edges in the manner we have, this bound is inclusive. Thus with seven colors, the set of distances \([a, \sqrt{7}a/2]\) can be forbidden.

**Corollary 1.1:** If \( F \) is a finite subset of the plane with at least 3 points such that \( M(F)/m(F) \leq \sqrt{7}/2 \), then \( \chi_r(R^2,F) \) is no greater than 7.

**Corollary 1.2:** If \( F \) is a finite subset of the plane with at least 3 points, and \( M(F)/m(F) \leq \sqrt{7}/2 \), then \( |F| \leq 7 \).

**Theorem 2:** To forbid any set of distances contained in an interval \([a, (n-1)a\sqrt{3}/2]\), \( a>0 \), requires at most \( n^2 \) colors. \( n>1 \).

**Proof:** Construct a tile of hexagons of size \( n^2 \) by stacking \( n \) hexagons on top of each other (Figure 5) then making \( n \) rows of these hexagons, such that every other row is at the same height (Figure 6).

**Figure 5:** Hexagons stacked \( n \) high for \( n = 3 \).

**Figure 6:** \( n \) rows of \( n \) hexagons for \( n = 3 \).
These \( n \) rows of \( n \) hexagons can be stacked vertically to infinity (Figure 7) and the strips this creates can be stacked horizontally to infinity (Figure 8).

Figure 7: 3x3 hexagon tile being stacked vertically.

Figure 8: 3x3 hexagon strips stacked horizontally.
Thus for any $n>0$, there is a tile on the plane composed of $n^2$ hexagons. By symmetry, each hexagon in the tile will have the same distances separating it from other hexagons of the same color in adjacent tiles.

Each hexagon is separated from the nearest hexagons of the same color in its column by the heights of $(n-1)$ hexagons. Therefore, the distance between closest hexagons of the same color in the same column is $(n-1)a\frac{\sqrt{3}}{2}$ where $a$ is the radius of the hexagon. The figure on the next page shows that this is the same distance for all hexagons in adjacent tiles. This is because these hexagons also have a stack of $n-1$ hexagon heights between them.

**Figure 9:** Tiles of hexagons with lines showing the distances between nearest hexagons of the same color.

**Corollary 2.1:** If $F$ is a finite subset of the plane with at least 3 points such that $M(F)/m(F) \leq (n-1)\frac{\sqrt{3}}{2}$, then $\chi_r(R^2,F)$ is no greater than $n^2$. 

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**Corollary 2.2:** If $F$ is a finite subset of the plane with at least 3 points, and $M(F)/m(F) \leq (n-1)\frac{\sqrt{3}}{2}$, then $F$ can have no more than $n^2$ points.

### 3 Other Theorems

**Theorem 3:** To forbid any set of distances contained in an interval $[a, (n(1.5)-1)a]$, $a>0$, requires at most $3n^2$ colors, $n>1$.

The proof of this theorem uses techniques similar to those in the proofs already shown. The tile constructed for this result is composed of $n$ rows of $3n$ hexagons in each row. All $3n^2$ hexagons in this quasi-rectangular array are colored differently. This coloring is repeated in each “column”, $n$ hexagons across, with the colorings in side-by-side columns shifted so as to maximize the distance between hexagons of the same color.

![Figure 10](image)  
**Figure 10:** Construction of tile for Theorem 3 for $n=2$.  

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**Corollary 3.1:** If $F$ is a finite subset of the plane with at least 3 points such that $M(F)/m(F) \leq n(1.5)-1$, then $\chi_r(R^2,F)$ is no greater than $3n^2$.

**Corollary 3.2:** If $F$ is a finite subset of the plane with at least 3 points, and $M(F)/m(F) \leq n(1.5)-1$, then $F$ can have no more than $3n^2$ points.

**Theorem 4:** To forbid any set of distances contained in an interval $[a, \sqrt{3 \ast (n^2 - n + \frac{1}{3})/2a}]$, $a>0$, requires at most $n^2 + 2n$ colors, $n>0$.

The tile constructed for this plane is composed of $n$ rows of $n+2$ hexagons.

**Figure 11:** Construction of tile for Theorem 4 for using $n = 3$. Due to lack of distinguishable colors, I am leaving the middle rows blue, but they should each be rows of $n+2$ colors.

**Corollary 4.1:** If $F$ is a finite subset of the plane with at least 3 points such that $M(F)/m(F) \leq \sqrt{3 \ast (n^2 - n + \frac{1}{3})/2}$, then $\chi_r(R^2,F)$
is no greater than \( n^2 + 2n \).

**Corollary 4.2:** If \( F \) is a finite subset of the plane with at least 3 points, and \( M(F)/m(F) \leq \sqrt{3 \ast (n^2 - n + \frac{1}{3})/2} \), then \( F \) can have no more than \( n^2 + 2n \) points.

**Theorem 5:** To forbid any set of distances contained in an interval \([a, (\sqrt{9 \ast n^2 - 3 \ast n + 1/2})a]\), \( a>0 \), requires at most \( 1+2n+2\sum_{k=1}^{n}(n+k) = 3n^2+3n+1 \) colors, \( n>0 \).

\( 3n^2+3n+1 \) is the number of regular hexagons appearing in the configuration consisting of a central hexagon wrapped in \( n \)-layers of hexagons.

**Figure 12:** \( n=2 \)
**Corollary 5.1:** If $F$ is a finite subset of the plane with at least 3 points such that $M(F)/m(F) \leq \sqrt{9 \cdot n^2 - 3 \cdot n + 1} / 2$, then $\chi_r(\mathbb{R}^2, F)$ is no greater than $3n^2 + 3n + 1$.

**Corollary 5.2:** If $F$ is a finite subset of the plane with at least 3 points, and $M(F)/m(F) \leq \sqrt{9 \cdot n^2 - 3 \cdot n + 1} / 2$, then $F$ can have no more than $3n^2 + 3n + 1$ points.

4 Conclusion:

Theorems 1-5 and their corollaries are obtained by considering tilings of the plane with tiles made up of congruent regular hexagons. Obviously there is no end to this method of obtaining rainbow coloring results for the plane, and we have more such results, but we deem it wise to withhold these for now. They will be more sensibly presented when systems have been developed for verifying that an arrangement of congruent regular hexagons constitute a tile for the plane.

References
