1. Say that $\sum_{n=0}^{\infty} c_n 2^n$ converges but $\sum_{n=0}^{\infty} c_n 4^n$ diverges. Circle the statements that we know must be true. (You do not need to show any work for this problem.)

Solution. I have bolded the statements we can assume above and crossed out the ones we cannot. We can think of this as a power series $\sum_{n=0}^{\infty} c_n x^n$ that converges for $x = 2$ and diverges for $x = 4$. Thus our interval of convergence for this power series is at least $(-2, 2)$ and is at most $[-4, 4)$. In particular, this power series must converge for $x = 1$ and diverge for $x = -5$. We cannot say for $x = 3$ and $x = -2$.

2. Consider the power series $\sum_{n=1}^{\infty} \frac{(2x - 4)^n}{n^2}$

(a) Find the center of this power series.

Solution. Solving for the center, we have:

$$2x - 4 = 0$$

$$x = 2$$

is our center. We can also find this from part (b), as we’ll see in a second.

(b) Find the interval of convergence of this power series.

Solution. Always use the ratio test to start these problems: We need

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(2x - 4)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x - 4)^n} \right| = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} \cdot |2x - 4| = 1 \cdot |2x - 4| < 1$$

and so the ratio test tells us that this series converges when

$$|2x - 4| < 1 \iff -1 < 2x - 4 < 1 \iff 3 < 2x < 5 \iff \frac{3}{2} < x < \frac{5}{2}$$

But we still need to test our endpoints!

If $x = 3/2$, we have: $\sum_{n=1}^{\infty} \frac{(2 \cdot \frac{3}{2} - 4)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges by the alternating series test (or by the $p$-series test).

If $x = 5/2$, we have: $\sum_{n=1}^{\infty} \frac{(2 \cdot \frac{5}{2} - 4)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which also converges by the $p$-series test.

Therefore our interval of convergence is $[3/2, 5/2]$. (Notice that the center will always be the average of the endpoints.)
3. Assume that \(|x| < 1\) for this question.

(a) Find a power series representation for \(f(x) = \frac{3x}{1 + x^5}\).

Solution. We will use the fact that if \(|x| < 1\), then

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n
\]

So then we can replace \(x\) in this equation above with \(-x^5\) and we get:

\[
\frac{1}{1 + x^5} = \frac{1}{1 - (-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n}
\]

So then our power series for \(f(x)\) is

\[
f(x) = \frac{3x}{1 + x^5} = 3x \cdot \frac{1}{1 + x^5} = 3x \sum_{n=0}^{\infty} (-1)^n x^{5n} = \sum_{n=0}^{\infty} 3(-1)^n x^{5n+1}
\]

for \(|x| < 1\). \(\triangle\)

(b) Use part (a) to integrate \(\int \frac{3x}{1 + x^5} \, dx\). You should leave your answer as a power series.

Solution. Integrating power series is really easy. We just integrate each term. We know how to integrate \(x^k\)—it’s just \(\frac{x^{k+1}}{k+1}\). With this in mind, we have

\[
\int \frac{3x}{1 + x^5} \, dx = \int \left( \sum_{n=0}^{\infty} 3(-1)^n x^{5n+1} \right) \, dx = \sum_{n=0}^{\infty} 3(-1)^n \int x^{5n+1} \, dx = \sum_{n=0}^{\infty} 3(-1)^n \frac{x^{5n+2}}{5n+2} + C
\]

(Remember, the +C is outside of this sum.) Integrating power series is great! :-) \(\triangle\)