I have not checked this review sheet for errors, hence there maybe errors in this document. thank you.

Class test II Review
sections 3.7(differentials)-5.5(logarithmic differentiation) excluding section 5.3(compound interest) is the syllabus.

SECTION 3.7

Question 1. Find the increment in \( x \) as \( x \) changes from \((a)\) 3 to 3.2, and \((b)\) when it changes from 3.2 to 3.

Solution. The increment in \( x \), \( \Delta x \) is always equal to the final value of \( x \) minus the initial value of \( x \). \((a)\) \( \Delta x = 3.2 - 3 = .2 \), and \((b)\) \( \Delta x = 3 - 3.2 = -.2 \).

Question 2. Find the increment in \( y \), \( \Delta y \) for the function \( f(x) = x^3 \), when \( x \) changes from 2 to 2.01.

Solution. \( \Delta y = f(2.01) - f(2.0). \) \( \Delta y = (2.01)^3 - 2^3 = 8.1206 - 8 = 0.1206 \).

Question 3. For the function \( f(x) = x^2 + 1 \).
(a). what is the differential in \( x \), \( dx \).
(b). What is the differential in \( y \), \( dy \).
(c). What is \( \Delta y \), \( dy \), \( dx \) when \( x \) changes from 2 to 2.01.
(d). and when \( x \) changes from 2.01 to 2.

Solution. (a). the differential in \( x \), \( dx \) is same as the increment in \( x \), \( \Delta x \). \( dx = \Delta x \)
(b). The general formula for \( dy \) is \( dy = f'(x) dx \). in this question \( dy = 2x dx \).
(c). \( f(2.01) = 5.0401, f(2) = 5 \). Hence when \( x \) changes from 2 to 2.01, \( \Delta y = 5.0401 - 5 = .0401 \). to use the formula for \( dy = 2x dx \), we have \( x = 2, dx = .01 \).
So when \( x \) changes from 2 to 2.01, \( dy = 2(2)(.01) = .04 \).

NOTE: \( dy \) is not equal to \( \Delta y \). \( dy \) is an approximation to the actual change, \( \Delta y \).
\( dx = 2.01 - 2 = .01 \).
(d). \( \Delta y = -.0401, dy = -.04, dx = -.01 \).

For a fixed \( x \), \( dy = f'(x) dx \), is a linear function of \( dx \). for example if starting value of \( x = 2 \), for the function \( x^2 + 1 \), \( dy = 2(2) dx = 4 dx \). For example if \( x \) changes from 2 to 2.05, \( dy = 4(.05) = .2 \). and when \( x \) changes from 2 to 1.99, \( dy = 4(-.01) = -.04 \).

Question 4. Find the differentials of the following functions.
(a). \( f(x) = \sqrt{3x^2 - x} \)
(b). \( f(x) = 2x^{3/2} + x^{1/2} \)
Solution. the differential of a function \( f(x) \) is \( df(x) = dy = f'(x) \, dx \)
(a). \( dy = \frac{6x-1}{2\sqrt{3x^2-x}} \, dx \).
(b). \( dy = (3x^{1/2} + \frac{1}{2\sqrt{x}}) \, dx \).

Question 5. Let \( f \) be the function defined by \( y = f(x) = x^2 + 1 \).
(a). Using differentials find the approximate change in \( y \), as \( x \) changes from 2 to 2.01.
(b). what is the actual change in \( y \) as \( x \) changes from 2 to 2.01.

Solution. The value of \( dy \), when \( x = 2 \), and \( dx = \Delta x = .01 \), gives an approximation to the actual change in \( y \), \( \Delta y \), as \( x \) changes from 2 to 2.01. From the solution to question 3 above we get \( dy = .04 \).
(b). The actual change in \( y \), when \( x \) changes from 2 to 2.01 is \( \Delta y = f(2.01) - f(2) = .0401 \).

Question 6. Use differentials to find the approximate value of the following numbers.
(a). \( \sqrt{99.7} \)
(b). \( \sqrt[3]{7.8} \)
(c). \( \sqrt[4]{81.6} \)
(d). \( \sqrt[3]{0.0096} \)

Solution. (a). \( \sqrt{99.7} = \sqrt{100-0.3} \) the answer should be near \( \sqrt{100} \). we use differentials to estimate \( \sqrt{99.7} - \sqrt{100} \). and use that to find an approximation to \( \sqrt{99.7} \). here \( y = \sqrt{x}, x = 100, dx = -.3. dy = \frac{1}{2\sqrt{x}} \, dx \). hence \( dy = \frac{-\frac{3}{20}}{2} = -.015 \). hence \( \sqrt{99.7} \sim 10 - .015 = 9.985 \).
(b). \( \sqrt[3]{7.8} = \sqrt[3]{8} - 0.2 \). so \( y = \sqrt[3]{x}, x = 8, dx = -0.2, dy = \frac{1}{3x^{2/3}} \, dx \), \( dy = -\frac{0.2}{3.37} = -0.0167 \). hence \( \sqrt[3]{7.8} \sim 2 - 0.0167 = 1.9833 \).
(c). \( \sqrt[4]{81.6} = \sqrt[4]{81} + .06 \). \( y = \sqrt[4]{x}, x = 81, dx = 0.6, \sqrt[4]{81.6} \sim 3 + \frac{0.6}{4(27)} = 3.0056 \).
(d). \( y = \sqrt[3]{x}, x = 0.001, dx = -0.00004 \). hence \( \sqrt[3]{0.00096} = 0.1 - \frac{0.00004}{3(0.01)} = 0.1 - 0.002 = 0.098 \)


Solution. Please refer to the back of the textbook, for the solution.

4.1

Question 1. Questions 5,6,7,8 Page 258.

Solution. (5). Increasing in \( (0,2) \), and decreasing in \( (\infty,0) \cup (2,\infty) \).
(6). Increasing in \( (1,0) \cup (1,\infty) \), and decreasing in \( (\infty,1) \cup (0,1) \).
(7). Increasing in \( (\infty,-1) \), and decreasing in \( (\infty,-1) \cup (1,\infty) \).
(8). Increasing in \( (\infty,-1) \cup (-1,\infty) \), and decreasing nowhere.

Question 2. Find the intervals where the function is increasing, and the intervals where the function is decreasing.
(a) \( \sqrt{16 - x^2} \).
(b) \( \frac{x^2}{x-1} \).
(c) \( x^3 - 3x^2 \).
(d) \( \frac{1}{4}x^3 - 3x^2 + 9x + 20 \).
(e) \( (x - 5)^{2/3} \).

**Solution.**

(a) the domain of \( f(x) \) is \([-4, 4]\). \( f'(x) = \frac{-2x}{2\sqrt{16 - x^2}} \). we need to find all \( x \), such that \( f'(x) \), is either undefined, or is equal to 0. \( f'(x) = \frac{-2x}{2\sqrt{16 - x^2}} \) is not defined when \( \sqrt{16 - x^2} = 0 \) or \( 16 - x^2 = 0 \), the solutions are \( x = 4, -4 \). the points where \( f'(x) = 0 \) is when \(-2x = 0 \), the solutions are \( x = 0 \). hence the points are \{ -4, 0, 4 \}. But since the domain of \( f(x) \) is \([-4, 4]\), the points divide the domain of \( f(x) \) into the intervals \([-4, 0)(0, 4)\). to find the sign of \( f'(x) \) in these intervals, we evaluate \( f'(x) \) at \(-1, 1\) respectively. we get \( f'(-1) = 1/\sqrt{15}, f'(1) = -1/\sqrt{15} \). hence the sign diagram of \( f'(x) \) is 

\[
-4 \quad + \quad 0 \quad - \quad 4
\]

hence \( f(x) \) is increasing in \([-4, 0)\), and decreasing in \((0, 4]\).

(b) \( f'(x) = \frac{x^2 - 2x}{(x-1)^2} \). the points where \( f'(x) \) is undefined is \( (x - 1)^2 = 0 \). the solutions are \( x = 1 \). the points where \( f'(x) = 0 \) are \( x^2 - 2x = 0 \), the solutions are \( x = 0, x = 2 \). hence the points are \( 0, 1, 2 \). the domain of \( f(x) \) is broken up by these numbers into the intervals \((-\infty, 0), (0, 1), (1, 2), (2, \infty)\). the sign diagram of \( f'(x) \) for these intervals is 

\[
-\infty \quad + \quad 0 \quad - \quad 1 \quad - \quad 2 \quad + \quad \infty
\]

Hence \( f(x) \) is increasing in the intervals \((-\infty, 0) \cup (2, \infty)\), and decreasing in the intervals \((0, 1) \cup (1, 2)\).

(c) \( f'(x) = 3x^2 - 6x \). \( f'(x) \) is a polynomial, hence it is defined everywhere. it is equal to zero in \( x = 0, x = 2 \). the sign diagram of \( f'(x) \) is 

\[
-\infty \quad + \quad 0 \quad - \quad 2 \quad + \quad \infty
\]

So \( f(x) \) is increasing in \((-\infty, 0) \cup (2, \infty)\), and decreasing in \((0, 2)\).

(d) \( f'(x) = x^2 - 6x + 9 \). \( f'(x) \) is always defined, and it is equal to zero at \( x = 3 \), so the sign diagram of \( f'(x) \) is 

\[
-\infty \quad + \quad 3 \quad + \quad \infty
\]

hence \( f(x) \) is increasing in \((-\infty, 3) \cup (3, \infty)\).

(e) \( f'(x) = \frac{2}{3(x-5)^{1/3}} \). \( f'(x) \) does not have any zero, and it is not defined at \( x = 5 \). so the sign diagram of \( f'(x) \) is \(-\infty \quad - \quad 5 \quad + \quad \infty\). So \( f(x) \) is decreasing in \((-\infty, 5)\), and increasing in \((5, \infty)\).

**Question 3.** Questions 42, 43, 44 in page 259.

**Solution.**

(42). \( r.\max = \text{none}, r.\min = 0 \) at \( x = -1 \).

(b) \( r.\max = \text{none}, r.\min = 2 \), at \( x = 0 \).

(44). \( r.\max = -9/2 \) at \( x = -3 \), \( r.\min = 9/2 \), at \( x = 3 \). Note the relative max is smaller than the relative min.

Solution. 45-(a). 46-(c).

Question 5. Using the first derivative test find the relative maxima, and the relative minima of the following functions.
(a). \( x^{5/3} \)
(b). \( (1/2)x^4 - x^2 \)
(c). \( (1/3)x^3 - x^2 - 3x + 4 \)
(d). \( \frac{x}{x^2 - 1} \)
(e). \( 2x^2 + \frac{4000}{x} + 10 \)
(f). \( x^{2/3} \)

Solution. (a). first we have to find the critical points. \( f'(x) = (5/3)x^{2/3} \). \( f'(x) \) is defined everywhere, and is zero only when \( x = 0 \). and \( x = 0 \) is in the domain of \( f(x) \). So 0 is the only critical point. the sign diagram of \( f'(x) \) is

\[
\begin{array}{ccccccccc}
-\infty & + & 0 & + & \infty
\end{array}
\]

So \( f'(x) \) does not change sign as we move across \( x = 0 \). So \( x = 0 \) is not a relative extrema.

(b). \( f'(x) = 2x^3 - 2x \). \( f'(x) \) is always defined, and is equal to 0, when \( x = -1, 0, 1 \). all these points are in the domain of \( f \). So each of them is a critical point. The sign diagram of \( f'(x) \) is

\[
\begin{array}{ccccccccc}
-\infty & - & -1 & + & 0 & - & 1 & + & \infty
\end{array}
\]

So we have relative maxima at \( x = 0 \), and relative minima at \( x = -1, 1 \). So \( f(0) = 0 \) is a relative max, and \( f(1) = -1/2, f(-1) = -1/2 \) are relative minima.

(c). relative min : \( f(3) = -5 \), relative max : \( f(-1) = 17/3 \).

(d). \( f'(x) = \frac{-1-x^2}{(x^2-1)^2} \). \( f'(x) \) doesnot have any zeroes since the numerator doesnot have any zeroes. \( f'(x) \) is not defined at \( x = 1, -1 \). But \( x = 1, -1 \) are not in the domain of \( f(x) \), hence these are not the critical points. Since \( f(x) \) doesnot have any critical points, it doesnot have any relative max, or relative min.

(e). \( f'(x) = 4x - 4000/x^2 \). \( f'(x) \) is not defined at \( x = 0 \). but \( x = 0 \) is not in the domain of \( f(x) \). So \( x = 0 \) is not a critical number. the zeroes of \( f'(x) \) are solutions of \( 4x - 4000/x^2 = 0 \), these are \( x = 10 \). Since \( x = 10 \) is in the domain of \( f(x) \), it is a critical number. The sign diagram of \( f'(x) \) is

\[
\begin{array}{ccccccccc}
-\infty & - & 0 & - & 10 & + & \infty
\end{array}
\]

Hence \( f'(x) \) changes sign from - to + as we move across \( x = 10 \). So \( f(10) = 610 \) is a relative minimum.

(f). \( f'(x) = \frac{2}{3x^{2/3}} \). \( f'(x) \) doesnot have any zero, and it is not defined when \( x = 0 \). and \( x = 0 \) is in the domain of \( f(x) \). So \( x = 0 \) is a critical number. The sign diagram of \( f'(x) \), is

\[
\begin{array}{ccccccccc}
-\infty & - & 0 & + & \infty
\end{array}
\]

. Hence \( x = 0 \) is a relative min.
4.2

**Question 1.** Questions (2), (4), (5), (8) on page-276.

**Solution.**

(2). cdown: (0, 1), cdown: (−∞, −4) ∪ (4, ∞).

(4). cdown: (−4, 4), cdown: (−∞, −4) ∪ (4, ∞).

(5). cdown: (0, 1), cdown: (−∞, 0) ∪ (1, ∞).

(8). cdown: (−∞, 0), cdown: (0, ∞).

**Question 2.** Question 10 on page-277.

**Solution.**

(a). \( f''(3) < 0, f''(5) > 0 \).

(b). \( f''(7) = 0 \) (inflexion point), \( f''(9) \) does not exist, as there is a corner at this point.

**Question 3.** Questions-11,12,13,14 on page-277.

**Solution.**

11-(a), 12-(b), 13-(b) (c does not have an inflexion point at \( x = 3 \), since \( x = 3 \) is not in the domain of \( f(x) \)), 14-(c).

**Question 4.** Questions 17-20 page 278.

**Solution.**

17-(c), 18-(a), 19-(d), 20-(b).

**Question 4.5.** what is the domain of (a). \( x^{1/3} \), (b). \( x^{1/2} \), (c). \( x^{4/3} \), (d). \( x^{5/2} \), (e). \( x^{-1/3} \), (f). \( x^{-1/2} \).

**Solution.**

(a). \( (−∞, ∞) \), (b). \([0, ∞) \), (c). \( (−∞, ∞) \), (d). \([0, ∞) \), (e). \( (−∞, 0) \cup (0, ∞) \), (f). \((0, ∞) \).

**Question 5.** Determine where the graph of the function is concave up, and where the graph is concave down, and find the inflexion points on the graph.

(a). \( x^4 - 6x^3 + 2x + 8 \)

(b). \( x^{4/7} \)

(c). \( \sqrt{\frac{1-x}{x}} \)

(d). \( \frac{x}{x+1} \)

(e). \( x + \frac{1}{x^2} \)

(f). \( (x-2)^{4/3} \)

**Solution.** \( f''(x) = 12x^2 - 36x \). \( f''(x) \) is defined every where and has the zeroes \( x = 0, x = 3 \). using these points we get the sign diagram of \( f''(x) \), \( −∞ \), \( 0 \), \( 3 \), \( −∞ \). hence \( f(x) \) is cup at \((−∞, 0) \cup (3, ∞) \), and cdown in \((0, 3) \). Also \( f''(x) \) changes sign at \( x = 0 \), and at \( x = 3 \), and the graph has a tangent corresponding to these points [you can find this by looking at it’s graph in the calculator], so \( (0, 8), (3, −67) \) are the inflexion points.

(b). \( f''(x) = \frac{-12}{49x^{10}} \). \( f''(x) \) does not have any zero, and it is defined at all points except at \( x = 0 \). So the sign diagram for \( f''(x) \), is \( −∞ \), \( 0 \), \( −∞ \). So the graph is cdown in \((−∞, 0) \cup (0, ∞) \), and cup nowhere. \((0, 0) \) is not an inflexion point as \( f''(x) \) does not change sign as we move across it.
(c). \( f''(x) = \frac{-1}{(4x^2 - 3x + 2)^2} \). It is defined for \( x < 4 \), and is nowhere 0, hence the sign diagram for \( f''(x) \) is \(-\infty -\infty -\infty 4\). So the graph is cup in \((-\infty, 4)\).

(d). \( f''(x) = \frac{-2}{(x+1)^2} \). \( f''(x) \) is never 0, and is not defined at \( x = -1 \), hence the sign diagram of \( f''(x) \) is \(-\infty -\infty -\infty -\infty -\infty -\infty \). So the graph is cup in \((-\infty, -1)\), and cdowm in \((-1, \infty)\). There is no inflexion point at \( x = -1 \), since the function is not defined at \( x = -1 \), or the graph has a hole at \( x = -1 \).

(e). \( f''(x) = \frac{6}{x^2} \). \( f''(x) \) is never 0, and it is undefined at \( x = 0 \). So the sign diagram for \( f''(x) \) is \(-\infty -\infty 0 + \infty \). Hence \( f(x) \) is cup in \((-\infty, 0) \cup (0, \infty)\), and cdowm nowhere. Since \( f(0) \) is not defined, \((0, f(0))\) is not an inflexion point, hence no inflexion points.

(f). \( f''(x) = \frac{4}{(x-2)^2} \). \( f''(x) \) is nowhere 0, and it is undefined at \( x = 2 \). hence the sign diagram for \( f''(x) \) is \(-\infty -\infty + 2 + \infty \). Hence \( f(x) \) is cup in \((-\infty, 2) \cup (2, \infty)\), and it is nowhere cdowm. Since \( f''(x) \) doesn't change sign as we move across \((2, 0)\), this point is not an inflexion point.

**Question 6.** Use the second derivative when applicable, to find the relative extrema of the given functions.

(a). \( \frac{1}{3}x^3 - 2x^2 - 5x - 5 \)

(b). \( t + \frac{9}{t} \)

**Solution.**

(a). \( f'(x) = x^2 - 4x - 5 \). The critical points are \( x = 5, x = -1 \). \( f''(x) \) exists at both these points, so we can use the 2nd derivative test to test each of the critical points for relative extrema. \( f''(x) = 2x - 4 \), and \( f''(5) = 2, f''(-1) = -6 \). So \((5, f(5))\) is a rmin, and \((-1, f(-1))\) is a rmax.

(b). \( f'(x) = 1 - 9/t^2 \). The critical points are \( x = 3, x = -3 \). at these points \( f''(x) \) is defined hence we can use 2nd derivative test to test these points for relative extrema. \( f''(-3) = -2/3 \), so \((-3, f(-3)) = \text{isanrelativemaxima, and} f''(3) = 3/34 \), so \((3, f(3))\) is a rmin.

**Question 7.** Questions 77, 79, 81 on page-280.

**Solution.** The solutions to these questions are available at the back of the book.

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4.3

**Question 1.** Questions 3, 5, 7, 9 on page-292.

**Solution.** back of the book.

**Question 2.** Questions 13,15,17,19,23,25

**Solution.** The solutions are available at the back of the textbook. The general idea is: for horizontal asymptotes, we find the limits \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). if any of these limit exist, there is a corresponding horizontal asymptote. hence a graph can have at the most two horizontal asymptotes. For vertical asymptotes, we find points \( x = a \), such that \( \lim_{x \to a^+} f(x) \) or \( \lim_{x \to a^-} f(x) \) is \( \infty \) or \( -\infty \). The graph has vertical asymptotes at these points. For (reduced) rational functions such points are exactly the points where the denominator is equal to 0. We still have to
make sure that the function has the right kind of limit at this point.

**Question 3.** Question 29, page-293.

**Solution.** at the back of the book.

4.4

There are two types questions in problems involving finding the absolute maxima/ minima of a function.

1. Finding the absolute extrema of a continuous function on a closed interval. In this case the absolute maxima and the abs minima exist. to find them, we find the value of the function at the boundary points of the given interval, and at the critical points, inside the given interval. then we choose the maximum and minimum values among these values.

To give an example of the last point. consider the question of finding the absolute extrema of \( f(x) = \frac{x}{x^2+2} \) in \([-1, 2]\). \( f(x) \) is continuous on the given interval. \( f'(x) = \frac{2-x^2}{(x^2+2)^2} \). \( f'(x) \) exists everywhere, since the denominator is always positive. \( f'(x) = 0, \) when \( 2 - x^2 = 0, \) at \( x = \sqrt{2}, -\sqrt{2} \). but \( -\sqrt{2} \approx -1.4 \) is not in the given interval, so it is not a critical number in the given interval so we ignore it. \( f(\sqrt{2}) = \sqrt{2}/4, f(-1) = -1/3, f(2) = 2/6. \) so the absolute max in the given interval is \( \sqrt{2}/4, \) and the absolute min is \(-1/3\).

2. Finding the absolute extrema of a function on a set which is not a closed interval. In this case, the absolute max/min may not exist. in most cases by looking at the graph(on a calculator or otherwise), we guess which points may be absolute extrema/minima. to find the exact location of these points, we in most cases choose the appropriate critical point, which we can find by finding the zeroes of \( f'(x) \) and the points where \( f'(x) \) does not exist.

**Question 1.** Questions (2), (3), (6), (8) in page-306.

**Solution.** (2). abmax 1/2, abmin -1/2 .
(3). abmax does not exist, abmin 0 .
(6) abmax doesnot exist, abmin doesnot exist.
(8). abmax = 1, abmin = -3 .

**Question 2.** Find the absolute max and the absolute min, if any for each of the following functions.
(a). \( 2x^2 + 3x - 4 \)
(b). \( x^{1/3} \)
(c). \( x^2 - 2x - 3 \) on \([-2, 3]\)
(d). \( x^3 + 3x^2 - 1 \) on \([-3, 2]\)
(e). \( \frac{x+1}{x-1} \) on \([2, 4]\)
(f). \( \frac{1}{x} \) on \((0, \infty)\)
(g). \( \frac{x}{x^2 + 2} \) on \([-1, 2]\)

Solution. (a). we have to find the ab. extrema over the domain which in this case is \((-\infty, \infty)\). From the graph \(f(x)\) doesnot have a absolute max, but it has an absolute min at a point where the slope of the tangent is 0. to find the point we solve \(f'(x) = 4x + 3 = 0\), which gives the solution \(x = -3/4\). hence \(f(-3/4) = -82/16\) is the absolute min.

(b). From the graph of \(x^{1/3}\) we can see that, it Doesnot have an ab. max or ab. min over it’s domain.

c). \(f'(x) = 2x - 2\). the critical numbers in \([-2, 3]\) is \(x = 1\). \(f(1) = -4, f(-2) = 5, f(3) = 0\). So the ab. max = 5, and the ab. min = -4.

d). \(f'(x) = 3x^2 + 6x\). The critical numbers in \([-3, 2]\) are \(x = 0, -2\). \(f(0) = -1, f(-2) = 3, f(-3) = -1, f(2) = 19\). the ab. max is 19, ab. min is -1.

e). in this case \(f(x)\) is continuous on \([2, 4]\), since \(x = 1\), is not in \([2, 4]\). \(f'(x) = \frac{-3}{(x-1)^2}\). No critical numbers in \([2, 4]\). \(f(2) = 3, f(4) = \frac{5}{3}\). So ab. max is 3, and ab. min is 5/3.

(f) from the graph we can see that \(1/x\) Doesnot have ab. extrema on \((0, \infty)\).

g). \(f'(x) = \frac{2-x^2}{(x^2 + 2)^2}\). The critical numbers in \([-1, 2]\) are \(x = \sqrt{2}\). \(f(\sqrt{2}) = \sqrt{2}/4, f(-1) = -1/3, f(2) = 2/6\). so the absolute max in the given interval is \(\sqrt{2}/4\), and the absolute min is -1/3.

4.5

Question 1. Find the dimensions of the rectangle with the largest possible area, that has a perimeter of 100 ft.

Solution. Let the length , breadth of the rectangle by \(x, y\) respectively. \(A = xy, 2x + 2y = 100\)
\[\Rightarrow x + y = 50, \text{and } x \geq 0, y \geq 0. \text{So } 0 \leq x \leq 50\]
\[y = 50 - x, \text{ } A = x(50 - x) = 50x - x^2\]

We want to maximize \(A(x) = 50x - x^2\), over \([0, 50]\). the critical numbers of \(A(x)\) in the interval are \(x = 25\). \(A(25) = 625, A(0) = 0, A(50) = 0\). So the maximum possible area is 625, and we have it when the length is 25, and the breadth is 25.

Question 2. By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, an open box can be made. if the cardboard is 15 inch long and 8 inch wide. find the dimensions of the box that will yield the max volume.

Solution. let the length of each cut be \(x\). Then the length, breadth, height of the resulting box is \(15 - 2x, 8 - 2x, x\) respectively, and the volume \(V(X) = (15 - 2x)(8 - 2x)x = 120x - 46x^2 + 4x^3\). and \(0 \leq 2x \leq 8\). So we have to maximize \(V(x)\) on \([0, 4]\). The critical numbers of \(V(x)\) in \([0, 4]\) are \(x = 5/3\). \(f(5/3) = 2450/27, V(0) = 0, V(4) = 0\). So the maximum possible volume is 2450/7 obtained when the dimensions of the box are \(35/3, 14/3, 5/3\).

Question 3. What are the dimensions of a closed rectangular box which has a square cross-section and a capacity of 128 inch\(^3\), and is constructed using the least
amount of material.

**Solution.** Let the base length of the box be $x$, and the height be $y$. Then $x^2y = 128$, hence $y = 128/x^2$. The surface area is $A = 2x^2 + 4xy = 2x^2 + 4x(128/x^2) = 2x^2 + 512/x$. $x > 0$ is the only condition on $x$. Hence we have to minimize $A(x) = 2x^2 + 512/x$ over $(0, \infty)$. By looking at the graph of $A(x)$ (possibly using the calculator) we can see that $A(x)$ has a min at a point where the slope of the tangent is horizontal. We locate this point by setting $A'(x) = 4x - 512/x^2 = 0$.

The solution is $x = (128)^{1/3} \approx 5.04$. At this value of $x$, $y = 128/(5.04)^2 \approx 5.04$. So the dimensions of the box with the least surface area is 5.04, 5.04, 5.04 respectively.

5.1

**Question 1.** Evaluate/simplify the expression using the rules of combining exponents.

\begin{align*}
(1) & \quad \frac{2^7 \cdot 4^{-1.3}}{2^{0.5}} \\
(2) & \quad \left( \frac{8}{27} \right)^{-1/3} \left( \frac{81}{256} \right)^{-1/4} \\
(3) & \quad \frac{(x^{2-3}y^{-3}z)^{1/2}}{(x^{2n-2}y^{2n})^{1/3}} \\
(4) & \quad \frac{x^{2n-2}y^{2n}}{x^{n+1}y^n}
\end{align*}

**Solution.**

\begin{align*}
(1) & \quad 4^{1.8} \\
(2) & \quad 2 \\
(3) & \quad \frac{64x^6}{y^4} \\
(4) & \quad x^{-(n+1)}y^n
\end{align*}

**Question 2.** (a). For the function $2^x$, what is the domain and range.

(b). Every function of the type $b^x$, ($b > 0$) passes through which point.

(c) On what interval is the function $2^x$ increasing and on what interval is the function decreasing.

(d) Same as (c) for the function $(0.5)^x$.

(e). what is $\lim_{m \to \infty} (1 + \frac{1}{m})^m$.

**Solution.**

(a). Domain is $(-\infty, \infty)$, range is $(0, \infty)$.

(b) $(0, 1)$

(c). Increasing in $(0, \infty)$, and decreasing nowhere.

(d) it is increasing nowhere and decreasing on $(-\infty, \infty)$.

(e). $e \approx 2.712$.

**Question 3.** Solve the following equations.

\begin{align*}
(1) & \quad 3^{3x-4} = 27 \\
(2) & \quad 8^x = \left( \frac{1}{32} \right)^{x-2} \\
(3) & \quad 3^{x^2} = \frac{1}{9} \\
(4) & \quad 9^x = 12.3^x + 27
\end{align*}

**Solution.**

\begin{align*}
(1) & \quad 3x - 4 = 3. \text{ so } x = 7/3 \\
(2) & \quad (2^3)^x = (2^{-5})^{x-2}. \text{ } \rightarrow \text{ } 3x = -5x + 10. \text{ So } x = 5/4.
\end{align*}
\[ (3) \quad x - x^2 = -2x. \text{ So } x = 0, x = 3 \]
\[ (4) \quad (3x)^2 - 12.3x + 27 = 0. \text{ Substituting } y = 3x, \text{ we have the equation } y^2 - 12y + 27 = 0, \text{ which has the solutions } y = 9, y = 3, \text{ hence } 3x = 9, 3x = 3. \]
\[ \text{Hence } x = 2, 1. \]

**Question 4.** Sketch the graph of \(2^x, (0.5)^x, 4^x, (0.25)^x\) on the same coordinate axes.

**Solution.** Use the calculator or Wolfram Alpha to check your answer.

### 5.2

If \(f(x) = \log_b(x)\) then \(b > 0, b \neq 1\).

**Question 1.**
1. What is the domain of \(\log_2(x)\), what is it’s range.
2. When is \(\log_2 x\) increasing, and when is it decreasing.
3. Same as part (2) but for the function \(\log_{0.5} x\).
4. Through which point does the graph of \(\log_b x\) pass through for every \(b > 0, b \neq 1\).

**Solution.**
1. Domain is \((0, \infty)\), range is \((-\infty, \infty)\).
2. \(\log_2 x\) is increasing on \((0, \infty)\) and decreasing nowhere.
3. \(\log_{0.5} x\) is increasing nowhere and decreasing on \((0, \infty)\)
4. \((1, 0)\)

**Question 2.** Evaluate
1. \(\log_{10} 100\)
2. \(\log_{10} 1\)
3. \(\log_{10} 10\)
4. \(\log_2 1\)
5. \(10^{\log_{10} x}\)
6. \(\log_{10}(0.01)\)
7. \(\log 10\)
8. \(\ln e\)
9. \(\log_{10}(10^x)\)

**Solution.**
1. 2
2. 0
3. 1
4. 0
5. \(x\)
6. \(-2\)
7. 1
8. 1
9. \(x\)

**Question 3.** Sketch the graph of \(\log_2 x, \log_{0.5} x\) on the same axes.

**Solution.** Please use Wolfram Alpha, to verify your answer.
Question 4. \( \log 3 = 0.4771 \), and \( \log 2 = 0.3010 \). Find \( \log \sqrt[3]{4} \) and \( \log 2 \).

Solution. \( \frac{3}{4} = \log 3 - 2 \log 2 = 0.4771 - 0.6020 = -0.1249 \).
\( \log \sqrt[3]{3} = \frac{1}{2} \log 3 = 0.23855 \).
\( \log \frac{1}{300} = -(\log 3 + 2) = -2.4771 \).

Question 5. Write as a single logarithm \( \ln 2 + \left( \frac{1}{2} \right) \ln(x + 1) - 2 \ln(1 + \sqrt{x}) \)

Solution. \( \ln \sqrt{x+1} (x+1)^2 \)

Question 6. Expand and simplify
(a). \( \log \frac{x+1}{x^2+1} \)
(b). \( \ln \frac{x^2}{x+1} \)
(c). \( \ln \frac{x^2(x+1)^3}{(x+2)^3} \)
(d). \( \ln \frac{x^2}{\sqrt{x(1+x)^2}} \)

Solution. (a). \( (1/2) \log(x + 1) - \log(x^2 + 1) \)
(b). \( x - \ln(1 + e^x) \)
(c). \( 2 \ln x + 3 \ln(x + 1) - 4 \ln(x + 2) \)
(d). \( 2 \ln x - (1/2) \ln x - 2 \ln(1 + x) \).

Question 7. Sketch the graph of \( y = e^{3x}, y = (1/3) \ln x \) on the same axes.

Solution. Please use wolfram alpha to verify your answer.

Question 8. Solve the equation.
(1) \( 2e^{-0.2t} - 4 = 6 \)
(2) \( \frac{50}{1 + 4^{0.2t}} = 20 \)

Solution. (1). \( e^{-0.2t} = 5 \), \( -0.2t = \ln 5 \). hence \( t = -\ln 5 / 0.2 \).
(2) \( e^{0.2t} = 3/8 \). \( 0.2t = \ln(3/8) \). \( t = \ln(3/8)/0.2 \).

5.4

Question 1. Find the derivative of the following functions.
(1) \( e^{f(x)} \)
(2) \( e^x \)
(3) \( e^{-x} \)
(4) \( e^x/x \)
(5) \( e^{-1/x} \)
(6) \( (4 - e^{-3x})^3 \)
(7) \( e^{x^2+2x+1} \)

Solution.
(1) \( e^{f(x)} f'(x) \)
(2) \( e^x \)
(3) \( -e^{-x} \)
(4) \( xe^x - e^x \)
(5) \( e^{-1/x} \left( \frac{1}{x^2} \right) \)
(6) $9(4 - e^{-3x})^2 e^{-3x}$
(7) $e^{x^2 + 2x + 1}(2x + 2)$

**Question 2.** Find an equation for the tangent line to the graph of $y = e^{2x - 3}$ at the point $(3/2, 1)$.

**Solution.** $y' = e^{2x - 3}(2)$. $y'(3/2) = 2$. The equation of the tangent is $y - 1 = 2(x - 3/2)$.

**Question 3.** Find the intervals where the function $f(x) = e^{-x^2/2}$ is increasing and where it is decreasing.

**Solution.** $f'(x) = -rac{x}{e^{x^2/2}}$. $f'(x)$ is defined everywhere since $e^{x^2/2}$ is never 0. $f'(x) = 0$, when $x = 0$. So the sign diagram of $f'(x)$ is

$\begin{array}{cccc}
-\infty & + & 0 & - \\

\end{array}$

So $f(x)$ is increasing on $(-\infty, 0)$, and decreasing on $(0, \infty)$.

$e^{\text{anything}} > 0$, so the zeros of $e^{\text{anything}}g(x)$ are same as the zeros of $g(x) = 0$. and to determine the sign of $e^{f(x)}g(x)$ we determine the sign of $g(x)$, since $e^{f(x)}$ does not affect the sign or the zeroes.

**Question 4.** Determine the intervals of concavity for the graph of the function $f(x) = xe^x$. Also find the inflexion points of the graph if any.

**Solution.** $f'(x) = e^x + xe^x = (x + 1)e^x$. $f''(x) = e^x + (1 + x)e^x = (2 + x)e^x$. $f''(x)$ is always defined, and $f''(x)$ has as zeroes $x = -2$. So the sign diagram of $f''(x)$ is

$\begin{array}{cccc}
-\infty & - & -2 & + \\

\end{array}$

So the graph is concave up in $(-2, \infty)$, and it is concave down in $(-\infty, -2)$. At the point $(-2, f(-2))$, the graph has a tangent, since $f'(-2)$ exists, and the concavity changes as we move across this point, so this point is an inflexion point of the graph.

**Question 5.** Questions 42, 43, 45 in sec-5.4 of our textbook.

**Solution.** are avaliable at the back of the book.
(2) \( \frac{1}{x} \)
(3) \( \frac{1}{x \ln x} \)
(4) \( \frac{\ln 2}{3x^2 - 2x + 1} \)
(5) \( (\ln(x + 1) - \ln(x - 1))' = \frac{1}{x + 1} - \frac{1}{x - 1} \)
(6) \( (3 \ln(u - 2))' = \frac{3}{u - 2} \)
(7) \( ((1/2) \ln(x^2 - 4))' = (1/2) \cdot \frac{2x}{x^2 - 4} \)

**Question 2.** Find the derivative of the following functions.

(1) \( y = (x - 1)^2(x + 1)^2(x + 3)^4 \)
(2) \( y = \frac{(2x^2 - 1)^5}{\sqrt{x^2 + 1}} \)
(3) \( \sqrt{\frac{3x^2}{x^2 + 1}} \)
(4) \( b^x \) for \( b > 0, b \neq 1 \)
(5) \( y = x^{x + 2} \)
(6) \( x^{\ln x} \)

**Solution.**

(1) \( \ln y = 2 \ln(x - 1) + 3 \ln(x + 1) + 4 \ln(x + 3) \)
\[ y' = \frac{2}{x - 1} + \frac{3}{x + 1} + \frac{4}{x + 3} \]
\[ y' = (x - 1)^2(x + 1)^2(x + 3)^4 \left( \frac{2}{x - 1} + \frac{3}{x + 1} + \frac{4}{x + 3} \right) \]

(2) \( \ln y = 5 \ln(2x^2 - 1) - (1/2) \ln(x + 1) \)
\[ y' = \frac{20x}{2x^2 - 1} - (1/2) \cdot \frac{1}{x + 1} \]
\[ y' = \left( 2x^2 - 1 \right)^{\frac{5}{2}} \left( \frac{20x}{2x^2 - 1} - (1/2) \cdot \frac{1}{x + 1} \right) \]

(3) \( \ln y = (1/2) \ln(4 + 3x^2) - (1/3) \ln(x^2 + 1) \)
\[ \text{using logarithmic differentiation} \]
\[ y' = \frac{\sqrt{4 + 3x^2}}{2x^2 + 1} \left( \frac{3x}{4 + 3x^2} - \frac{2x}{3(x^2 + 1)} \right) \]

(4) Using logarithmic differentiation \( y' = b^x \ln b \)

(5) \( \ln y = (x + 2) \ln x \)
\[ y' = (x^{x + 2})(\ln x + \frac{x + 2}{x}) \]

(6) \( \ln y = (\ln x)(\ln x) = (\ln x)^2 \)
\[ y' = x \cdot \ln x \left( \frac{2 \ln x}{x} \right) \]

**Question 3.** Use implicit differentiation to find \( y' \), when \( \ln(xy) - y^2 = 5 \)

**Solution.** \( \ln x + \ln y - y^2 = 5 \)
\[ \frac{1}{x} + \frac{y'}{y} - 2yy' = 0 \]
\[ y' = \frac{-1}{y - 2y} \]

**Mock exam I**

All question numbers refer to the review material above. The solutions of these questions are provided above.

**Question 1.** [20pts] Sec 3.7 6-d.

**Question 2.** [20pts] Sec 4.1 5-e.
Question 3. [20pts] Sec 5.5 2-3.

Question 4. [20pts] Sec 4.4 2-(d).

Question 5. [20pts] Sec 4.2 5-(f).

Question 6. [10pts] Sec 4.3 15.

Mock exam II

All question numbers refer to the review material above. The solutions of these questions are provided above.

Question 1. [20pts] Sec 3.7 6-c.

Question 2. [20pts] Sec 4.1 2-d.

Question 3. [20pts] Sec 5.5 2-5.

Question 4. [20pts] Sec 4.4 2-(g).

Question 5. [20pts] Sec 4.2 5-(e).

Question 6. [10pts] Sec 4.3 19.

Mock exam III

All question numbers refer to the review material above. The solutions of these questions are provided above.

Question 1. [20pts] Sec 3.7 6-a.

Question 2. [20pts] Sec 5.4 4.

Question 3. [20pts] Sec 5.5 2-1.

Question 4. [20pts] Sec 4.5 2.

Question 5. [20pts] Sec 4.1 5-(b).

Question 6. [10pts] Sec 4.3 23.