Problem. The Snake Lemma: Consider the following commutative diagram, which is exact at $B$ and $B'$.

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\delta} & 0 \\
\downarrow{u} & & \downarrow{v} & & \downarrow{w} & & \\
0 & \xrightarrow{\alpha'} & A' & \xrightarrow{\beta'} & B' & \xrightarrow{\gamma} & C'
\end{array}
$$

Prove that there exists an exact sequence

$$
\begin{array}{cccc}
Ker(u) & \xrightarrow{\pi} & Ker(v) & \xrightarrow{\beta|_{Ker(v)}} & Ker(w) & \xrightarrow{\delta} & Coker(u) & \xrightarrow{\pi'} & Coker(v) & \xrightarrow{\beta'} & Coker(w)
\end{array}
$$

Where $\pi = \alpha|_{Ker(u)}$, $\beta = \beta|_{Ker(v)}$, and $\pi'$, $\beta'$ are defined by

- $\pi'(a' + \text{im}(u)) = \alpha'(a') + \text{im}(v)$ for all $a' \in A'$
- $\beta'(b' + \text{im}(v)) = \beta'(b') + \text{im}(w)$ for all $b' \in B'$.

Solution. To define $\delta$, suppose $x \in Ker(w)$. Since $\beta$ is surjective, $\exists y \in B$ such that $\beta(y) = x$. By the commutativity of the diagram, we have that $\beta v(y) = w \beta(y) = 0$. So $v(y) \in Ker(\beta')$. Since the lower sequence is exact at $B'$, $\exists z \in A'$ with $\alpha'(z) = v(y)$. Note that $\alpha'$ is injective and so $z$ is unique. Let $\delta(x) := z + \text{im}(u)$. It is not clear that this map is well-defined. That is, that the choice of the pre-image of $x$ does not change our choice of coset of $\text{im}(u)$.

Suppose $y, \tilde{y} \in B$ such that $\beta(y) = x = \beta(\tilde{y})$. Let $z, \tilde{z} \in A'$ such that $\alpha'(z) = v(y)$ and $\alpha'(\tilde{z}) = v(\tilde{y})$. Since $\beta(y - \tilde{y}) = \beta(y) - \beta(\tilde{y}) = x - x = 0$, and the top sequence is exact at $B$, $\exists a \in A$ such that $\alpha(a) = y - \tilde{y}$. Therefore

$$
\alpha'(z - \tilde{z}) = v(y - \tilde{y})
= v\alpha(c)
= a'v(c)
$$

Since $\alpha'$ is injective, we conclude that $u(c) = z - \tilde{z}$. I.e. that $z - \tilde{z} \in \text{im}(u)$. So $z + \text{im}(u) = \tilde{z} + \text{im}(u)$, and we have that $\delta$ is well-defined.

We must also verify that $\pi$ and $\beta$ have the promised codomains. Suppose $a \in Ker(u)$. Then $v\alpha(a) = a'u(a) = \alpha'(0) = 0$. So $\pi(a) = \alpha(a) \in Ker(v)$. Now, given $b \in Ker(v)$, $w\beta(b) = \beta'v(b) = \beta'(0) = 0$. So $\beta(b) = \beta(b) \in Ker(w)$.

Now for a procession of tedious verifications:

1. $\text{im}(\pi) = Ker(\beta')$: Let $b \in \text{im}(\pi)$. Then $b = \pi(a)$ for some $a \in Ker(u)$. So $\beta(b) = \beta\pi(a) = \beta\alpha(a) = 0$. Thus $\text{im}(\pi) \subseteq Ker(\beta')$.

Let $c \in Ker(\beta')$. Then $0 = \beta(c) = \beta(c)$. So $c \in Ker(\beta) \cap Ker(v)$. By the exactness of the top sequence at $B$, $c \in \text{im}(\alpha)$. Let $a \in A$ be such that $\alpha(a) = c$. Then

$$
\begin{align*}
\alpha'v(a) &= \pi(a) \\
&= v(c) \\
&= 0.
\end{align*}
$$
But $\alpha'$ is injective, so $u(a) = 0$. Thus $a \in \text{Ker}(u)$. Then $c = \alpha(a) = \pi(a)$. I.e. $c \in \text{im}(\pi)$. So $\text{Ker}(\overline{\beta}) \subseteq \text{im}(\pi)$. Hence $\text{im}(\pi) = \text{Ker}(\overline{\beta})$.

2. $\text{im}(\overline{\beta}) = \text{Ker}(\delta)$: Let $c \in \text{im}(\overline{\beta})$. Then $c = \overline{\beta}(b) = \beta(b)$ for some $b \in \text{Ker}(v)$. As was noted in the construction of $\delta$, $\exists z \in A'$ such that $\alpha'(z) = v(b) = 0$. Since $\alpha'$ is injective, $z = 0$. So $\delta(c) = z + \text{im}(u) = 0 + \text{im}(u)$. I.e. $c \in \text{Ker}(\delta)$. So $\text{im}(\overline{\beta}) \subseteq \text{Ker}(\delta)$.

Now suppose $c \in \text{Ker}(\delta)$. Then $\exists b \in B$ such that $\beta(b) = c$. As was noted in the constructions of $\delta$, $\exists z \in A'$ such that $\alpha'(z) = v(b)$. Since $0 + \text{im}(u) = \delta(c) = z + \text{im}(u)$, we have $z \in \text{im}(u)$. Thus $\exists y \in A$ such that $u(y) = z$. Then

$$v(\alpha(y)) = \alpha'(u(y))$$
$$= \alpha'(z)$$
$$= v(b).$$

Therefore $b - \alpha(y) \in \text{Ker}(v)$, and

$$\overline{\beta}(b - \alpha(y)) = \beta(b - \alpha(y))$$
$$= \beta(b) - \beta\alpha(y)$$
$$= \beta(b)$$
$$= c$$

So $c \in \text{im}(\overline{\beta})$ and $\text{Ker}(\delta) \subseteq \text{im}(\overline{\beta})$. Hence $\text{im}(\overline{\beta}) = \text{Ker}(\delta)$.

3. $\text{im}(\delta) = \text{Ker}(\overline{\alpha'})$: Let $a' + \text{im}(u) \in \text{im}(\delta)$. Then $\exists c \in \text{Ker}(w)$ and $b \in B$ such that $\beta(b) = c$ and $\alpha'(a') = v(b)$. Therefore

$$\overline{\alpha'}(a' + \text{im}(u)) = \alpha'(a') + \text{im}(v)$$
$$= v(b) + (\text{im})(v)$$
$$= 0 + \text{im}(v).$$

So $a' + \text{im}(u) \in \text{Ker}(\overline{\alpha'})$. Thus $\text{im}(u) \subseteq \text{Ker}(\overline{\alpha'})$.

Now let $a' + \text{im}(u) \in \text{Ker}(\overline{\alpha'})$. Then $\alpha'(a') \in \text{im}(v)$. So $\exists b \in B$ such that $\alpha'(a') = v(b)$. Since $v(b) \in \text{im}(\alpha')$, and the lower sequence is exact at $B'$, $\beta'(v(b)) = 0$. Then $w(\beta(b)) = \beta'(v(b)) = 0$. So $\beta(b) \in \text{Ker}(w)$. Therefore we have $\delta(\beta(b)) = a' + \text{im}(u)$. So $a' + \text{im}(u) \in \text{im}(\delta)$, and $\text{Ker}(\overline{\alpha'}) \subseteq \text{im}(\delta)$. Hence $\text{im}(\delta) = \text{Ker}(\overline{\alpha'})$.

4. $\text{im}(\overline{\alpha'}) = \text{Ker}(\overline{\beta'})$. Let $b' + \text{im}(v) \in \text{im}(\overline{\alpha'})$. Then $\exists a' + \text{im}(u) \in \text{Coker}(u)$ such that

$$b' + \text{im}(v) = \overline{\alpha'}(a' + \text{im}(u))$$
$$= \alpha'(a') + \text{im}(v)$$

Thus $\exists b \in B$ such that $b' - \alpha'(a') = v(b)$.

$$\overline{\beta'}(b' + \text{im}(v)) = \beta'(b') + \text{im}(w)$$
$$= (\beta' v(b) + \beta' \alpha'(a')) + \text{im}(w)$$
$$= (w \beta(b) + 0) + \text{im}(w)$$
$$= 0 + \text{im}(w).$$
Now let $b' + \text{im}(v) \in \text{Ker}(\overline{\beta'})$. Then $\beta'(b') \in \text{im}(w)$. So $\exists c \in C$ such that $w(c) = \beta'(b')$. Since $\beta$ is surjective, $\exists b \in B$ such that $\beta(b) = c$. Therefore
\[
\beta'(v(b)) = w\beta(b) = w(c) = \beta'(b').
\]

So $b' - v(b) \in \text{Ker}(\beta')$. By the exactness of the lower sequence at $B'$, $\exists a' \in A'$ such that $\alpha'(a') = b' - v(b)$. So
\[
\overline{\alpha'}(a' + \text{im}(u)) = \alpha'(a') + \text{im}(v) = (b' - v(b)) + \text{im}(v) = b' + \text{im}(v).
\]

Thus $b' + \text{im}(v) \in \text{im}(\overline{\alpha'})$ and $\text{Ker}(\overline{\beta'}) \subseteq \text{im}(\overline{\alpha'})$. Hence $\text{im}(\overline{\alpha'}) = \text{Ker}(\overline{\beta'})$.

And thus we see that the sequence is exact.

Now further assume that $\alpha$ is injective. Suppose $\overline{\alpha}(a) = 0$ for $a \in \text{Ker}(\overline{u})$. Then $\overline{\alpha}(a) = 0$. So $a = 0$. Thus $\overline{\alpha}$ is injective as well.

Assume now that $\beta'$ is surjective. Let $c' + \text{im}(w) \in \text{Coker}(w)$. Since $\beta'$ is surjective, $\exists b' \in B'$ such that $\beta'(b') = c'$. Then
\[
\overline{\beta'}(b' + \text{im}(v)) = \beta'(b') + \text{im}(w) = c' + \text{im}(w).
\]

Thus $\overline{\beta}$ is surjective.

**Problem.** Let $R := \mathbb{C}[X, Y, Z] / (X^2 + Y^2 + Z^2 - 1)$, and $x, y, z$ denote the image of $X, Y, Z$ respectively. Let $\pi : R^3 \to R$ be defined by $\pi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax + by + cz$. Prove that $K := \text{Ker}(\pi)$ is a free $R$–module.

**Solution.** Define $j : R \to R^3$ by $j(r) := \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$. Then $\pi j(r) = rx^2 + ry^2 + rz^2 = r$. So $j$ is a splitting map.

Thus $R^3 \simeq K \oplus j(R) = K \oplus \text{span} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. To show that $K$ is free, it suffices to show that there are vectors $u, v$ such that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $u, v$ form a basis form $R^3$. $\begin{pmatrix} -i \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ iz \\ x - iy \end{pmatrix}$ are such vectors, since
\[
\det \begin{pmatrix} x & -i & 0 \\ y & 1 & iz \\ z & 0 & x - iy \end{pmatrix} = x^2 + y^2 + z^2 = 1.
\]

We can construct an explicit basis for $K$ as follows. Suppose $\alpha$ is such that
\[
\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in K.
\]
Then
\[
0 = (x \ y \ z) \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (y - ix) - \alpha.
\]

So \( \alpha = (y - ix) \).

Now suppose \( \beta \) is such that
\[
\begin{pmatrix} 0 \\ iz \\ x - iy \end{pmatrix} - \beta \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in K.
\]

Then
\[
0 = (x \ y \ z) \begin{pmatrix} 0 \\ iz \\ (x - iy) \end{pmatrix} - \beta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = xz - \beta.
\]

So \( \beta = xz \).

Substituting the values of \( \alpha \) and \( \beta \) gives us an explicit basis of \( K \).

\[
\begin{pmatrix} ix^2 - xy - i \\ ixy - y^2 + 1 \\ ixz - yz \end{pmatrix}, \begin{pmatrix} -x^2z \\ iz - xyz \\ x - iy - xz^2 \end{pmatrix}.
\]