Problem (4). The Comparison Theorem For Injective Modules:

Consider the following diagram

\[
\begin{array}{rcccl}
B_\bullet: & 0 & \longrightarrow & M & \longrightarrow & B_0 \xrightarrow{\varphi_0} B_1 \xrightarrow{\varphi_1} B_2 \longrightarrow & \cdots \\
\downarrow{\pi} & & & \downarrow & & \downarrow & \cdots \\
Q_\bullet: & 0 & \longrightarrow & N & \longrightarrow & Q_0 \xrightarrow{\psi_0} Q_1 \xrightarrow{\psi_1} Q_2 \longrightarrow & \cdots \\
\end{array}
\]

where \( B_\bullet \) is exact and each \( Q_i \) is injective. Prove that there exists a chain map \( f : B_\bullet \rightarrow Q_\bullet \) lifting \( \pi \). Further, if \( g \) is another chain map lifting \( \pi \), then \( f \sim g \).

Solution. By the injectiveness of \( Q_0 \), \( \exists f_0 \) such that the following diagram commutes.

\[
\begin{array}{c}
M \xrightarrow{\psi_0} B_0 \\
\downarrow{\psi_0 \pi} \\
Q_0 \xrightarrow{\exists f_0}
\end{array}
\]

That is, \( \psi_0 \pi = f_0 \varphi_0 \).

Now inductively, assume that we have defined \( f_n : B_n \rightarrow Q_n \) such that \( \psi_n f_{n-1} = f_n \varphi_n \) for some \( n \geq 0 \) (For convenience define \( f_{-1} = \pi \)).

By the injectiveness of \( Q_{n+1} \), \( \exists f_{n+1} \) such that the following diagram commutes.

\[
\begin{array}{c}
B_n \xrightarrow{\varphi_{n+1}} B_{n+1} \\
\downarrow{\psi_{n+1} f_n} \\
Q_{n+1} \xrightarrow{\exists f_{n+1}}
\end{array}
\]

That is, \( \psi_{n+1} f_n = f_{n+1} \varphi_{n+1} \). By induction, \( \exists f_n : B_n \rightarrow Q_n \ \forall n \in \mathbb{N} \) that makes the diagram commute.

Now suppose \( g \) is a chain map lifting \( \pi \).

We seek homomorphisms \( s_n : B_n \rightarrow Q_{n-1} \) such that \( f_n - g_n = \psi_n s_n + s_{n+1} \varphi_{n+1} \).

Set \( s_0 := 0 \).

Note that

\[
\psi_0 \pi = f_0 \varphi_0 \\
\psi_0 \pi = g_0 \varphi_0
\]

Then \( (f_0 - g_0) \circ \varphi_0 = 0 \). So \( \ker(\varphi_1) = \text{im}(\varphi_0) \subseteq \ker(f_0 - g_0) \). Define the maps \( \widetilde{\varphi_1} \) and \( \widetilde{f_0 - g_0} \) by

\[
\begin{array}{c}
\widetilde{\varphi_1} : \frac{B_0}{\ker(\varphi_1)} \rightarrow B_1 \\
b + \ker(\varphi_1) \rightarrow \varphi_1(b)
\end{array}
\]

\[
\begin{array}{c}
\widetilde{f_0 - g_0} : \frac{B_0}{\ker(\varphi_1)} \rightarrow Q_0 \\
b + \ker(\varphi_1) \rightarrow (f_0 - g_0)(b)
\end{array}
\]
These maps are well-defined since $\text{ker}(\phi_1)$ is contained in both $\text{ker}(\phi_1)$ and $\text{ker}(f_0 - g_0)$. Note also that $\tilde{\varphi}_1$ is injective.

Thus we have the diagram

$$
\begin{array}{c}
0 & \rightarrow & \frac{B_0}{\text{ker}(\varphi_1)} & \rightarrow & \frac{B_1}{\text{ker}(\varphi_1)} \rightarrow & B_1 \\
\downarrow & & \tilde{\varphi}_1 & & \rightarrow & \\
\downarrow & & f_0 - g_0 & & \rightarrow & \\
Q_0 & & & & \rightarrow & \\
\end{array}
$$

By the injectivity of $Q_0$, $\exists s_1 : B_1 \rightarrow Q_0$ such that $f_0 - g_0 = s_1 \tilde{\varphi}_1$.

Now let $b \in B_0$.

$$(f_0 - g_0)(b) = (f_0 - g_0)(b + \text{ker}(\varphi_1))$$

$$= s_1 \tilde{\varphi}_1(b + \text{ker}(\varphi_1))$$

$$= s_1(\varphi_1(b))$$

$$= s_1\varphi_1(b).$$

Thus $f_0 - g_0 = s_1\varphi_1$.

Inductively assume that for all $1 \leq k \leq n$, we have defined $s_k : B_k \rightarrow Q_{k-1}$ such that $f_k - g_k = \psi_k s_k + s_{k+1} \varphi_{k+1}$.

Note that

$$(f_n - g_n)\varphi_n = f_n\varphi_n - g_n\varphi_n$$

$$= \psi_n f_{n-1} - \psi_n g_{n-1}$$

$$= \psi_n(f_{n-1} - g_{n-1})$$

$$= \psi_n(\psi_n - s_{n-1} + s_n \varphi_n)$$

$$= \psi_n s_n \varphi_n$$

Let $\gamma := f_n - g_n - \psi_n s_n$. Then $\text{ker}(\varphi_{n+1}) = \text{im}(\varphi_n) \subseteq \text{ker}(\gamma)$. Define functions $\tilde{\varphi}_{n+1}$ and $\tilde{\gamma}$ by

$$
\begin{array}{c}
\tilde{\varphi}_{n+1} : \frac{B_n}{\text{ker}(\varphi_{n+1})} \rightarrow B_{n+1} \\
B_n \rightarrow B_{n+1} \\
b + \text{ker}(\varphi_{n+1}) \rightarrow \varphi_{n+1}(b) \\
\tilde{\gamma} : \frac{B_n}{\text{ker}(\varphi_{n+1})} \rightarrow Q_n \\
B_n \rightarrow Q_n \\
b + \text{ker}(\varphi_{n+1}) \rightarrow \gamma b \\
\end{array}
$$

These maps are well-defined since $\text{ker}(\varphi_{n+1})$ is contained in both $\text{ker}(\varphi_{n+1})$ and $\text{ker}(\gamma)$. Note also that $\tilde{\varphi}_{n+1}$ is injective.

So we have the following diagram

$$
\begin{array}{c}
0 & \rightarrow & \frac{B_n}{\text{ker}(\varphi_{n+1})} & \rightarrow & \frac{B_{n+1}}{\text{ker}(\varphi_{n+1})} & \rightarrow & B_{n+1} \\
\downarrow & & \tilde{\varphi}_{n+1} & & \rightarrow & & \\
\downarrow & & \tilde{\gamma} & & \rightarrow & & \\
Q_n & & & & \rightarrow & & \\
\end{array}
$$

By the injectivity of $Q_n$, $\exists s_{n+1} : B_{n+1} \rightarrow Q_n$ such that $\tilde{\gamma} = s_{n+1} \tilde{\varphi}_{n+1}$. 

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Then for any \( b \in B_n \),

\[
(f_n - g_n)(b) - \psi_n s_n(b) = \gamma(b) \\
= \tilde{\gamma}(b + \text{ker}(\varphi_{n+1})) \\
= s_{n+1}\varphi_{n+1}(b + \text{ker}(\varphi_{n+1})) \\
= s_{n+1}\varphi_{n+1}(b).
\]

Thus \( f_n - g_n = \psi_n s_n + s_{n+1}\varphi_{n+1} \).

By induction, there are maps \( s_n \) satisfying the desired equations. Thus \( f \sim g \).
**Problem (7).** Let $\Gamma$ be an oriented graph with vertices $\{v_1, \cdots, v_n\}$ and edges $\{e_1, \cdots, e_m\}$ and $c$ connected components. Let $C_0$ be the free module on the vertices, $C_1$ be the free module on the edges, and $\partial$ be the incidence matrix of $\Gamma$. Consider the simplicial chain complex, $\Gamma_\bullet$, of $\Gamma$:

$$\Gamma_\bullet : 0 \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow 0$$

Compute $H_0(\Gamma_\bullet)$ and $H_1(\Gamma_\bullet)$.

**Solution.** We seek to compute the rank and nullity of $\partial$. Suppose $\partial^T w = 0$. Note that $\partial^T$ has exactly one 1 and one $-1$ in every row corresponding to an edge of $\Gamma$. Therefore if there is an edge from $v_i$ to $v_j$, we must have $w_i - w_j = 0$. So $w_i = w_j$. Suppose now that two vertices $v_i$ and $v_j$ lie in the same connected component. Then there is a path $(v_i, v_k_1, \cdots, v_k_s, v_j)$. But then by the above observation, $w_i = w_{k_1} = \cdots = w_{k_s} = w_j$. Therefore $w_i = w_j$ if $v_i$ and $v_j$ lie in the same connected component. But then we see that when choosing $w$ we had exactly $c$ choices of values. So $\text{nullity}(\partial^T) = c$. And by Rank-Nullity (to be safe I will assume that $R$ is a PID), $\text{rank}(\partial^T) = n - c$.

But $\text{rank}(\partial) = \text{rank}(\partial^T) = n - c$. So again by Rank-Nullity, $\text{nullity}(\partial) = m - (n - c) = m - n + c$.

Now we see that

$$H_0(\Gamma_\bullet) \cong \frac{C_0}{\text{im}(\partial)} \cong \frac{R^n}{R^{n-c}} \cong R^c$$

$$H_1(\Gamma_\bullet) \cong \text{ker}(\partial) \cong R^{m-n+c}$$

In particular, if $\Gamma$ is connected, $c = 1$ and the original statement of the problem follows.
Problem (12). Consider the following diagram of $R$-modules with exact rows

$$
\begin{array}{cccccccc}
A & \to & B & \to & C & \to & D & \to & E \\
\downarrow u & & \downarrow v & & \downarrow w & & \downarrow y & & \downarrow z \\
A' & \to & B' & \to & C' & \to & D' & \to & E'
\end{array}
$$

(a) If $v$ and $y$ are injective, and $u$ is surjective, then $w$ is injective.

(b) If $v$ and $y$ are surjective and $z$ is injective, then $w$ is surjective.

(c) If $u$, $v$, $y$, and $z$ are isomorphisms, then $w$ is an isomorphism.

Solution.

(a) Assume $v, y$ are injective, and $u$ is surjective. Suppose $w(c) = 0$. Then $y\gamma(c) = \gamma' w(c) = 0$. So $\gamma(c) \in \ker(y) = \{0\}$. Thus $c \in \ker(\gamma) = \text{im}(\beta)$. Therefore $\exists b \in B$ such that $\beta(b) = c$. Then

$$
\beta' v(b) = w \beta(b) = w(c) = 0.
$$

So $v(b) \in \ker(\beta') = \text{im}(\alpha')$. And so $\exists a' \in A'$ with $\alpha'(a') = v(b)$. Since $u$ is surjective, $\exists a \in A$ with $u(a) = a'$. Thus $v(b) = \alpha' u(a) = v \alpha(a)$. Since $v$ is injective, $b = \alpha(a)$. And so $c = \beta(b) = \beta \alpha(a) = 0$. Thus $w$ is injective.

(b) Assume $v, y$ are surjective, and $z$ is injective. Suppose $c' \in C'$. Since $y$ is surjective, $\exists d \in D$ such that $y(d) = \gamma'(c')$. So $0 = \delta' \gamma'(c') = \delta' y(d) = \gamma \delta(d)$. Since $z$ is injective, $\delta(d) = 0$. That is $d \in \ker(\delta) = \text{im}(\gamma)$. So $\exists c \in C$ with $\gamma(c) = d$. Then

$$
\gamma'(c') = y(d) = y \gamma(c) = \gamma' w(c)
$$

Therefore $c' - w(c) \in \ker(\gamma') = \text{im}(\beta')$. So $\exists b' \in B'$ with $\beta'(b') = c' - w(c)$. Since $v$ is surjective, $\exists b \in B$ with $v(b) = b'$. Then

$$
c' - w(c) = \beta'(b') = \beta' v(b) = w \beta(b)
$$

And so $c' = w(c + \beta(b))$, and $w$ is surjective.

(c) Assume $u, v, y$, and $z$ are isomorphisms. Then by part (a), $w$ is injective. By part (b) $w$ is surjective. Thus $w$ is an isomorphism.
Problem (13). For $\alpha \in A$, let $C_{\alpha}$ be the chain complex:

$$\ldots \overset{\partial_{n,i+2}}{\rightarrow} C_{\alpha,i+1} \overset{\partial_{n,i+1}}{\rightarrow} C_{\alpha,i} \overset{\partial_{n,i}}{\rightarrow} \ldots$$

Define the chain complex $\oplus_{\alpha} C_{\alpha}$ and prove that $H_{n}(\oplus_{\alpha} C_{\alpha}) \cong \oplus_{\alpha} H_{n}(C_{\alpha})$.

Solution. Define the chain complex $\oplus_{\alpha} C_{\alpha}$ by:

$$\ldots \overset{\oplus_{\alpha} \partial_{n,i+2}}{\rightarrow} \oplus_{\alpha} C_{\alpha,i+1} \overset{\oplus_{\alpha} \partial_{n,i+1}}{\rightarrow} \oplus_{\alpha} C_{\alpha,i} \overset{\oplus_{\alpha} \partial_{n,i}}{\rightarrow} \ldots$$

Where

$$(\oplus_{\alpha} \partial_{n,i}) ((x_{\alpha})_{\alpha \in A}) = (\partial_{n,i}(x_{\alpha}))_{\alpha \in A}.$$ 

$\oplus_{\alpha} C_{\alpha}$ is a chain complex.

Let $(x_{\alpha})_{\alpha \in A} \in \oplus_{\alpha} C_{\alpha,i+1}$. Then

$$(\oplus_{\alpha} \partial_{n,i} \circ \oplus_{\alpha} \partial_{n,i+1}) ((x_{\alpha})_{\alpha \in A}) = \oplus_{\alpha} \partial_{n,i} ((\partial_{n,i+1}(x_{\alpha}))_{\alpha \in A})$$

$$= ((\partial_{n,i} \partial_{n,i+1})(x_{\alpha}))_{\alpha \in A}$$

$$= (0)_{\alpha \in A}.$$ 

So $\oplus_{\alpha} C_{\alpha}$ is a chain complex.

Now let $n \in \mathbb{N}$. Define $\varphi : H_{n}(\oplus_{\alpha} C_{\alpha}) \rightarrow \oplus_{\alpha} H_{n}(C_{\alpha})$ by:

$$\varphi((x_{\alpha})_{\alpha \in A} + im(\oplus_{\alpha} \partial_{n+1})) = (x_{\alpha} + im(\partial_{n+1}))_{\alpha \in A}$$

$\varphi$ is well-defined: Suppose $(x_{\alpha})_{\alpha \in A} + im(\oplus_{\alpha} \partial_{n+1}) = (y_{\alpha})_{\alpha \in A} + im(\oplus_{\alpha} \partial_{n+1})$. Then

$$(x_{\alpha} - y_{\alpha})_{\alpha \in A} + im(\oplus_{\alpha} \partial_{n+1}) = im(\oplus_{\alpha} \partial_{n+1}).$$

$$\implies (x_{\alpha} - y_{\alpha})_{\alpha \in A} \in im(\oplus_{\alpha} \partial_{n+1}).$$

$$\implies x_{\alpha} - y_{\alpha} \in im(\partial_{n+1})$$

$$\implies x_{\alpha} + im(\partial_{n+1}) = y_{\alpha} + im(\partial_{n+1})$$

$$\implies (x_{\alpha} + im(\partial_{n+1}))_{\alpha \in A} = (y_{\alpha} + im(\partial_{n+1}))_{\alpha \in A}$$

$$\implies \varphi((x_{\alpha})_{\alpha \in A} + im(\oplus_{\alpha} \partial_{n+1})) = \varphi((y_{\alpha})_{\alpha \in A} + im(\oplus_{\alpha} \partial_{n+1})).$$

And so $\varphi$ is well-defined.

$\varphi$ is surjective: To see this let $(y_{\alpha} + im(\partial_{n+1}))_{\alpha \in A} \in \oplus_{\alpha} H_{n}(C_{\alpha})$. Then $y_{\alpha} \in ker(\partial_{n+1})$ for all $\alpha \in A$ and

$y_{\alpha} \in im(\partial_{n+1})$ for all but finitely many $\alpha$. That is, $\exists B \subseteq A$ with $|B| < \infty$ such that $y_{\alpha} \notin im(\partial_{n+1})$ for all $\alpha \in B$ and $y_{\alpha} \in im(\partial_{n+1})$ for all $\alpha \notin B$. Define

$$x_{\alpha} := \begin{cases} y_{\alpha} & \text{if } \alpha \in B \\ 0 & \text{if } \alpha \notin B. \end{cases}$$

Then $(x_{\alpha})_{\alpha \in A} \in \oplus_{\alpha} (C_{\alpha,n})$. Also, $(x_{\alpha})_{\alpha \in A} \in ker(\oplus_{\alpha} \partial_{n,n})$. Thus, $(x_{\alpha})_{\alpha \in A} + im(\partial_{n+1}) \in H_{n}(\oplus_{\alpha} C_{\alpha})$. Now,

$$\varphi((x_{\alpha})_{\alpha \in A} + im(\partial_{n+1})) = (x_{\alpha} + im(\partial_{n+1}))_{\alpha \in A}$$

$$= (y_{\alpha} + im(\partial_{n+1}))_{\alpha \in A}.$$
Therefore \( \varphi \) is surjective.

\( \varphi \) is injective: Let \((x_\alpha)_{\alpha \in A} + \text{im}(\partial_{\alpha,n+1}) \in H_n(\oplus_{\alpha} C_\alpha)\) and suppose that

\[
(x_\alpha + \text{im}(\partial_{\alpha,n+1}))_{\alpha \in A} = \varphi((x_\alpha)_{\alpha \in A} + \text{im}(\partial_{\alpha,n+1})) = (0 + \text{im}(\partial_{\alpha,n+1}))_{\alpha \in A}.
\]

Then \( x_\alpha \in \text{im}(\partial_{\alpha,n+1}) \) for all \( \alpha \). So for each \( \alpha \), \( \exists w_\alpha \in C_{\alpha,n+1} \) such that \( \partial_{\alpha,n+1}(w_\alpha) = x_\alpha \). Note that since finitely many of the \( x_\alpha \) are non-zero, finitely many of the \( w_\alpha \) are non-zero. So \((w_\alpha)_{\alpha \in A} \in \oplus(C_{\alpha,n+1})\), and

\[
(\oplus_{\alpha} \partial_{\alpha,n+1})(w_\alpha) = (\partial_{\alpha,n+1}(w_\alpha))_{\alpha \in A} = (x_\alpha)_{\alpha \in A}.
\]

Therefore \((x_\alpha)_{\alpha \in A} \in \text{im}(\partial_\alpha,n+1)\) and \( \varphi \) is injective.

Since \( \varphi \) is an isomorphism, \( H_n(\oplus_{\alpha} C_\alpha) \cong \oplus_{\alpha} H_n(C_\alpha) \).
Problem (15). The simplicial homology of a tetrahedron, $T$, is given by the homology of the following complex:

$$
0 \longrightarrow \mathbb{R}^4 \xrightarrow{\partial_2} \mathbb{R}^6 \xrightarrow{\partial_1} \mathbb{R}^4 \longrightarrow 0
$$

Where $\partial_1$ and $\partial_2$ are given by the following matrices:

$$
\partial_1 = \begin{pmatrix}
12 & 13 & 14 & 23 & 24 & 34 \\
1 & -1 & -1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & -1 & -1 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 & -1 \\
4 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{pmatrix}
$$

$$
\partial_2 = \begin{pmatrix}
123 & 124 & 134 & 234 \\
12 & 1 & 1 & 0 & 0 \\
13 & -1 & 0 & 1 & 0 \\
14 & 0 & -1 & -1 & 0 \\
23 & 1 & 0 & 0 & -1 \\
24 & 0 & 1 & 0 & -1 \\
34 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
$$

Compute the simplicial homology of $T$.

Solution. The following computations were done using a computer algebra system.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Rank</th>
<th>Nullity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial_1$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\partial_2$</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore:

$$
H_0(T) \cong \text{coker}(\partial_1) \cong \frac{\mathbb{R}^4}{\mathbb{R}^3} \cong \mathbb{R}
$$

$$
H_1(T) \cong \frac{\text{ker}(\partial_1)}{\text{im}(\partial_2)} \cong \frac{\mathbb{R}^3}{\mathbb{R}^3} \cong 0
$$

$$
H_2(T) \cong \text{ker}(\partial_2) \cong \mathbb{R}
$$