Math 831 Homework 1: Due February 4.

Throughout $R$ will denote a commutative ring with identity. You may discuss these problems with your classmates or me and you may also freely use results established or assumed in class. Please do not consult with other students, professors, texts or online material.

1. Let $P \subseteq R$ be an ideal. Show that the following are equivalent:
   
   (a) $P$ is a prime ideal.
   
   (b) For all ideals $I, J \subseteq R$, if $I \cdot J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

   Conclude that if $P$ is a prime ideal and $I \cap J \subseteq P$ for ideals $I, J \subseteq R$, then $I \subseteq P$ or $J \subseteq P$.

2. A multiplicatively closed set $S \subseteq R$ is said to be saturated if whenever $uv \in S$, then $u \in S$ and $v \in S$.

   Prove that $S$ is a saturated multiplicatively closed set if and only if $S$ is the complement of a union of prime ideals.

3. Let $S, T \subseteq R$ be multiplicatively closed sets. Show that $(R[S])[X]$ is isomorphic to $R[X]$. Here $ST$ denotes the multiplicatively closed set $\{st \mid s \in S$ and $t \in T\}$.

4. Let $a \in R$ be a non-zerodivisor and set $S := \{1, a, a^2, \ldots\}$. Prove that $R$ is an integral domain if and only if $R[S]$ is an integral domain.

5. Let $S \subseteq R$ be a multiplicatively closed set, $I \subseteq R$ an ideal and $R[X]$ the polynomial ring in one variable over $R$.

   (a) Prove that $R[S][X]$ is isomorphic to $R[X]_S$.
   
   (b) Prove that $R[X]/IR[X]$ is isomorphic to $(R/I)[X]$. Conclude that if $P$ is a prime ideal of $R$, then $PR[X]$ is a prime ideal of $R[X]$.
   
   (c) Let $M$ be a maximal ideal of $R$. Prove that $MR[X]$ is never a maximal ideal of $R[X]$.
   
   (d) Use (a) and (b) to prove that there cannot exist a chain of prime ideals $Q_1 \subset Q_2 \subset Q_3$ in $R[X]$ contracting to same prime ideal in $R$.
   
   (e) Can you generalize the statements (a)-(d) above to the polynomial ring in any finite set of indeterminates over $R$?

6. Let $a \in R$ be a non-nilpotent element and set $S := \{1, a, a^2, \ldots\}$.

   (a) Prove that $R_S$ is isomorphic to $R[X]/(aX - 1)$.
   
   (b) Use part (a) to show that if $p \in \mathbb{Z}$ is a prime number and $R$ denotes the set of rational numbers whose denominator is not divisible by $p$, then $(pX - 1)$ is a maximal ideal in $R[X]$.
   
   (c) Formulate and prove a version of part (a) for an arbitrary multiplicatively closed set $S \subseteq R$. For this, it is helpful to first prove that if $\phi : R \to T$ is a homomorphism of commutative rings such that $\phi(s)$ is a unit for all $s \in S$, then $\phi$ factors through $R_S$. 

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7. Let $I \subseteq R$ be an ideal and $S \subseteq R$ a multiplicatively closed set. Prove that $R_S/I_S$ is isomorphic to $(R/I)_S$, where $S$ denotes the image of $S$ in $R/I$. In other words, localization commutes with the formation of factor rings.

8. Let $X_1, \ldots, X_n$ be indeterminates over $R$. For a unit $u \in R$ and $f(X_2, \ldots, X_n) \in R[X_2, \ldots, X_n]$, set $X_1' := uX_1 + f_2(X_2, \ldots, X_n)$. Prove that $R[X_1, \ldots, X_n] = R[X_1', X_2, \ldots, X_n]$.

9. Let $R$ be an integral domain, $n \geq 2$, and $X_1, Y_1, \ldots, X_n, Y_n$ indeterminates over $R$. Prove that the ring

$$A := R[X_1, Y_1, \ldots, X_n, Y_n]/(X_1Y_1 + \cdots + X_nY_n)$$

is an integral domain. Conclude that if $K$ is a field and $X, Y, Z, W$ are indeterminates over $K$, then the ring $K[X, Y, Z, W]/(XY - ZW)$ is an integral domain.

10. Give a rigorous proof that if $K$ is a field and $X, Y, Z, W$ are indeterminates over $K$, then the ring $K[X, Y, Z, W]/(XY - ZW)$ is not a unique factorization domain.