Throughout $R$ will denote a commutative ring with identity and $M$ an $R$-module. You may discuss these problems with your classmates or me and you may also freely use results established or assumed in class. Please do not consult with other students, professors, texts or online material.

1. Use the Krull principal ideal theorem and the prime avoidance lemma to prove the following statement. Let $R$ be a Noetherian ring, and suppose there exist prime ideals $P_0 \subset P_1 \subset P_2$. Then there exist infinitely many prime ideals $P'$ satisfying $P_0 \subset P' \subset P_2$.

2. Prove that if $(R, m)$ is a local ring of Krull dimension $d$, then $R[[X]]$ is a local ring of Krull dimension $d + 1$. You may assume that $R[[X]]$ is a Noetherian ring.

3. Let $I \subseteq R$ be an ideal. Let $IM$ denote the submodule of $M$ generated by the set $\{i \cdot x \mid i \in I \text{ and } x \in M\}$.

   (i) Prove that there is a well-defined $R/I$ structure on $M/IM$. In particular, if $Q$ is a maximal ideal, then $M/QM$ is a vector space over the field $R/Q$.

   (ii) Let $\phi : M_1 \to M_2$ be an $R$-module homomorphism. Prove that there is a well-defined $R/I$-homomorphism $\phi' : M_1/IM_1 \to M_2/IM_2$, so that $\phi'(x + IM) = \phi(x) + IM$.

   (iii) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of $R$-modules. Discuss the exactness or failure thereof of the corresponding induced sequence of $R/I$-modules $0 \to A/IA \xrightarrow{\alpha'} B/IB \xrightarrow{\beta'} C/IC \to 0$.

4. Let $R$ be a Noetherian, local ring with maximal ideal $m$ and assume that $M$ is finitely generated. Let $X = \{x_1, \ldots, x_n\}$ be a generating set for $M$. Prove that $X$ is minimal among generating sets if and only if the images of the $x_i$ in $M/mM$ form a basis for $M/mM$ as a vector space over $R/m$. Conclude that every minimal generating set for $M$ has the same number of elements.

5. Assume that $R$ is a Noetherian ring and $0 \neq M$ is a finitely generated $R$-module. Prove that there exists an exact sequence of the form $R^t \xrightarrow{\phi} R^n \xrightarrow{\pi} M \to 0$, for some $t \geq 0$ and $n \geq 1$. If we regard the elements in $R^t$ and $R^n$ as column vectors, explain how we can think of the homomorphism $\phi$ as being given by an $n \times t$ matrix. Furthermore, if $R$ is a local ring with maximal ideal $m$, prove that $\pi(e_1), \ldots, \pi(e_n)$ is a minimal generating set for $M$ if and only if the entries of $\phi$ (when thought of as a matrix) belong to $m$. Here, $e_1, \ldots, e_n$ are the standard basis elements for $R^n$.

6. Let $(R, m)$ be a Noetherian local ring and assume $M$ is a finitely generated $R$-module. Prove that there exists an exact sequence of the form $\cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$, such that each $F_i$ is a finitely generated free $R$-module and each matrix $\phi_i$ has entries in $m$. Such a sequence is called a minimal free resolution of $M$.

7. A composition series of length $n$ for $M$ is a chain of submodules $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$. 

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such that each quotient $M_i/M_{i-1}$ is a non-zero simple $R$-module, i.e., it has no proper submodules. In particular, this means that for each $1 \leq i \leq n$, we cannot insert an extra submodule in the chain above between $M_{i-1}$ and $M_i$. Prove that if $M$ has a composition series, then any two composition series have the same length. Comment: This is the module analogue of the Jordan-Hölder theorem from group theory. In this setting, the proof is easier than the one for groups. To prove this version, induct on the minimal length of a composition series.

8. Prove that $M$ has a composition series if and only if $M$ is both Artinian and Noetherian.

9. If an $R$-module $W$ has a composition series, we say that $W$ has finite length. The length of $W$, denoted $\lambda(W)$, is the length of any composition series. Show that if $0 \to A \to B \to C \to 0$ is an exact sequence of $R$-modules, then $B$ has finite length if and only if $A$ and $C$ have finite length. If these conditions hold, prove that $\lambda(B) = \lambda(A) + \lambda(C)$.

10. Assume that $R$ is a Noetherian, local ring with maximal ideal $m$. Let $W_n$ denote the submodule of $M$ consisting of those elements annihilated by $m^n$. Prove that if $M$ is Artinian then each $W_i$ is finitely generated and $M = \bigcup_{n \geq 1} W_n$. (Note: $W_1$ is called the socle of $M$.)