1. For this we may localize at $P_2$ and mod out $P_0$. Thus we begin again assuming that $R$ is a local domain with maximal ideal $\mathfrak{m}$ having height at least two. We must see that there are infinitely many non-zero prime ideals not equal to $\mathfrak{m}$. Let $0 \neq a \in \mathfrak{m}$. Take $Q$ a minimal prime over the ideal $\langle a \rangle$. Since $R$ is an integral domain, height($Q$) = 1, by Krull’s principal ideal theorem. Suppose there were only finitely primes between $\mathfrak{m}$ and $(0)$, then there would be only finitely many primes $Q$ minimal over principal ideals of $R$. Call these primes $Q_1, \ldots, Q_r$. Then $\mathfrak{m} \subseteq Q_1 \cup \cdots \cup Q_r$. Therefore, $\mathfrak{m} \subseteq Q_i$, some $i$, by the prime avoidance lemma, and thus $\mathfrak{m} = Q_i$, a contradiction. Thus there must exists infinitely many primes non-zero primes not equal to $\mathfrak{m}$.

2. Let’s first note that $R[[X]]$ is a local ring with maximal ideal $\langle \mathfrak{m}, X \rangle$. This follows immediately from the fact that a power series is not a unit if and only if its constant term is not a unit, i.e., its constant term belongs to $\mathfrak{m}$. Since $\langle \mathfrak{m}, X \rangle$ is the set of power series with constant term in $\mathfrak{m}$, the set of non-units forms an ideal, and hence $R[[X]]$ is local with maximal ideal $\langle \mathfrak{m}, X \rangle$.

To calculate the dimension of $R[[X]]$, let $P \subseteq R$ be a prime ideal. It is not difficult to see that $P[[X]]$ is a prime ideal in $R[[X]]$, where for any ideal $J \subseteq R$, $J[[X]]$ denotes the set of all power series with coefficients in $J$. Thus, if $P_0 \subseteq \cdots \subseteq P_d = \mathfrak{m}$ is a maximal chain in $R$, then $P_0[[X]] \subseteq \cdots \subseteq m[[X]] \subseteq \langle \mathfrak{m}, X \rangle$ is a chain of length $d + 1$. This shows $R[[X]]$ has dimension at least $d + 1$. Let $a_1, \ldots, a_d$ be a system of parameters for $R$, i.e., $\mathfrak{m}$ is a minimal prime over $\langle a_1, \ldots, a_d \rangle$. If $Q \subseteq R[[X]]$ is a prime minimal over $\langle a_1, \ldots, a_d, X \rangle$, then $Q$ contains $\mathfrak{m}$ and $X$, and thus $Q = \langle \mathfrak{m}, X \rangle$. It follows that $\dim(R[[X]]) = \text{height}(\mathfrak{m}) \leq d + 1$, by the general form of Krull’s principal ideal theorem. Thus $R[[X]]$ has dimension $d + 1$.

3. For the first statement, one just has to show that the $R/I$ scalar multiplication on $M/IM$ given by $(r + I) \cdot (x + IM) = rx + IM$ is well-defined. Similarly, for part (ii), the key point is that $\phi'$ is well defined. For this, if $x + IM_1 = y + IM_1$, then $x - y \in IM_1$. This means $x - y = \sum_j i_j m_j$, where $i_j \in I$ and $m_j \in M_1$. Applying $\phi$ we obtain $\phi(x) - \phi(y) = \sum_j i_j \phi(m_j)$, which shows that $\phi(x) + IM_2 = \phi(y) + IM_2$, i.e., $\phi'$ is well defined.

For the third part regarding the induced exact sequence, it is straightforward to show that $\beta'$ is surjective and that the image of $\alpha'$ is contained in the kernel of $\beta'$. Suppose $\beta'(b') \equiv 0$ in $C/IC$, where $b' \in B/IB$ is the class of an element $b \in B$. Then $\beta(b) \in IC$. Thus, we may write $\beta(b) = \sum_i i_i c_i$, for finitely many $i_i \in I$ and $c_i \in C$. Since $\beta$ is surjective, each $c_i = \beta(b_i)$, for some $b_i \in B$. Therefore, $\beta(b) = \sum_i i_i \beta(b_i)$ and thus $b - \sum_i i_i b_i$ belongs the kernel of $\beta$ and hence the image of $\alpha$. It follows that we can write $\alpha(a) = b - \sum_i i_i b_i$, for some $a \in A$. But this means $\alpha'(a') \equiv b'$ in $B/IB$, so that the kernel of $\beta'$ is contained in the image of $\alpha'$.

It is not true in general that the induced map $\alpha'$ is injective. For example, take $R = \mathbb{Z}$, $A = B = \mathbb{Z}$, $C = \mathbb{Z}/2\mathbb{Z}$, and $I = 2\mathbb{Z}$. Let $\alpha$ be multiplication by 2 and $\beta$ be the canonical map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$.
Then for $0 \neq 1 + 2\mathbb{Z}$ in $A/IA$, $\alpha'(1 + 2\mathbb{Z}) \equiv 2 + 2\mathbb{Z} \equiv 0$ in $B/I$, so the induced map $\alpha'$ is not injective.

A parting comment: The induced complex obtained by modding out $I$ in this problem is a special case of obtaining a new complex from an original one by taking the tensor product of each module in the given short exact sequence with an auxiliary module. Just as in our case, exactness is preserved everywhere in the induced sequence except at the term on the left. This failure of exactness is measured by the so-called torsion modules, e.g., in our case, $\text{Tor}_1(C, R/I)$, and an initial study of such modules is on our horizon.

4. Suppose $X$ is minimal among generating sets for $M$. Then clearly the images of the elements in $X$ span the vector space $M/mM$. If these images were not linearly independent, then one of them, say $x_1 + mM$ could be deleted. Let $N$ denote the submodule of $M$ generated by $\{x_2, \ldots, x_n\}$. Since the images of $x_2, \ldots, x_n$ generate $M/mM$, $M = N + mM$, so by Nakayama’s Lemma, $N = M$, i.e., $M$ is generated by $\{x_2, \ldots, x_n\}$, a contradiction. Thus, the images of the elements of $X$ form a basis for $M/mM$. The converse is similar. For the second statement, it follows from the first statement that the number of elements in any minimal generating set for $M$ equals the dimension of the vector space $M/mM$.

5. If $M$ is generated $x_1, \ldots, x_n$, we may define a surjective $R$-module homomorphism $\pi$ from $R^n$ to $M$ by taking each column vector $(r_1, \ldots, r_n)^t$ to $r_1x_1 + \cdots + r_nx_n$. (Here $^t$ stands for transpose.) Since $R$ is Noetherian, the kernel of $\pi$ is finitely generated, say by $t$ elements. Then, in a similar fashion, we can define a surjective map $\phi$ from $R^t$ to the kernel of $\pi$, which gives the required exact sequence. The matrix representation of $\phi$ is just like for vector spaces over a field - the $j$th column of the matrix consists of the coefficients obtained when one expresses the value under $\phi$ of the $j$th standard basis element of $R^t$ in terms of the standard basis of $R^n$.

Now suppose $(R, m)$ is a local ring and $x_1 = \pi(e_1), \ldots, x_n = \pi(e_n)$ is a minimal generating set for $M$. Let $(r_1, \ldots, r_n)$ be a column of the matrix of $\phi$. Then $r_1x_1 + \cdots + r_nx_n = 0$. Thus, in $M/mM$, $\sum_{i=1}^n (r_i + m)(x_i + mM) \equiv 0$. By the previous problem, the $\{x_1 + mM, \ldots, x_n + mM\}$ are linearly independent, so we must have each $r_i + m \equiv 0$ in $R/m$, i.e., each $r_i \in m$. The converse is similar.

6. This is really just an elaboration of the previous problem. Maintaining the notation from the second part, we take $\pi$ as before, $F_0 = R^n$, $F_1 = R^t$, and $\phi i_1 = \phi$. The sequence obtained in the previous problem becomes $F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\pi} M \xrightarrow{0}$, where $\phi_1$ has entries in $m$. To continue, let $F_2$ be a free module whose rank equals the minimal number of generators of the kernel of $\phi_1$. The we map the standard basis of $F_2$ onto a fixed minimal generating set. Calling this map $\phi_1$ we now obtain an exact sequence $F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\pi} M \xrightarrow{0}$, where now both $\phi_1$ and $\phi_2$ have entries in $m$. We can now proceed by induction to construct all of the terms in the required sequence.

It should be noted that if at any stage in the process, the kernel of $\phi_n$ is free of rank $n$, then the map $\phi_{n+1} : F_{n+1} \rightarrow \text{ker}(\phi_n)$ is an isomorphism, and hence $\text{ker}(\phi_{n+1}) = 0$. Thus resolution must terminate in order to maintain minimality, and therefore we have an exact sequence

$$0 \xrightarrow{} F_{n+1} \xrightarrow{\phi_{n+1}} F_n \xrightarrow{\phi_n} \cdots \xrightarrow{} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\pi} M \xrightarrow{0}.$$  

7. Let $(0) \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ and $(0) \subseteq M'_1 \subseteq \cdots \subseteq M'_n = M$ be two composition series. We induct on the minimum of $\{n, r\} = n$ (say). If $n$ equals one, then $M$ is a simple module, and this
forces $r = 1$. Suppose $n > 1$. On the one hand,

$$(0) \subseteq M_2/M_1 \subseteq \cdots \subseteq M_n/M_1 = M/M_1$$

is a composition series for $M/M_1$ of length $n - 1$ (via one of the standard isomorphism theorems). Now consider $M_1 \cap M_1'$. If this intersection is non-zero, then it must equal $M_1 = M_1'$, since both $M_1$ and $M_1'$ are simple modules. This chain must terminate, since $M/M_1$ is both Artinian and Noetherian. Let $n > 1$.

We now also have that $M_1/M_1'$ is simple. For $2 \leq i \leq t - 1$, the quotient of two consecutive terms in the series above is

$$(M_i' + M_1)/M_1 \subseteq \cdots \subseteq (M_{i-1}' + M_1)/M_1 = M_i'/M_1 \subseteq \cdots \subseteq M_t'/M_1 = M/M_1$$

a simple $R$-module. For $i = t$, since $M_i' \cap M_1 \neq 0$, $M_i' \cap M_1 = M_1$, so $M_1 \subseteq M_t'$. On the other hand, $M_i' \cap M_1 = M_1$, so $M_i \subseteq M_i'$. Thus, $(M_i' + M_1)/M_1 = M_i'/M_1$. We now also have that $M_i \subseteq M_i'$ for $i \geq t$, and the rest of the claim follows from this and the usual isomorphism theorem.

8. Suppose $M$ has a composition series of length $n \geq 1$. We induct on $n$ to see that $M$ is both Artinian and Noetherian. If $n = 1$, $M$ has no proper submodules, so it is clearly Artinian and Noetherian. Suppose $n > 1$ and $M_1$ is a simple submodule of $M$ at the bottom of a composition series for $M$, so that $M_1$ is both Artinian and Noetherian. Then we have an exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$. Since $M/M_1$ has a composition series of length $n - 1$ (as in the previous problem), $M/M_1$ is both Artinian and Noetherian, by induction. Thus $M$ is both Artinian and Noetherian.

Now suppose that $M$ is both Artinian and Noetherian. Let $M_1$ be minimal among the non-zero submodules of $M$. Then $M_1$ exists, since $M$ is Artinian, and by definition, $M_1$ must be a simple $R$-module. Now, let $M_2$ be minimal among the submodules of $M$ properly containing $M_1$. Then $M_2/M_1$ is simple. Continuing in this fashion, we construct an ascending chain whose quotients are simple modules. This chain must terminate, since $M$ is Noetherian. If it terminates at $M_n$, then $M_n = M$, otherwise, there exists $M_{n+1}$ properly containing $M_n$ and we can shrink this module to one minimal over $M_n$, thereby contradicting stability of the chain.

9. The first statement follows from the previous problem and the corresponding statements for the Noetherian and Artinian properties. For the additivity of length, let $K$ denote the image of $A$ in $B$. Then $A$ is isomorphic to $K$, so $\lambda(K) = \lambda(A)$. Similarly, $B/K$ is isomorphic to $C$. 

so these modules have the same length. Thus, we must show that \( \lambda(K) + \lambda(B/K) = \lambda(B) \).
But now it is clear that if we take \((0) \subseteq K_1 \subseteq \cdots \subseteq K_r = K\) a composition series for \(K\) and \(K/K \subseteq B_2/K \subseteq \cdots \subseteq B_t/K = B/K\) a composition series for \(B/K\), then
\[
(0) \subseteq K_1 \subseteq \cdots \subseteq K_r = K \subseteq K_1 \subseteq \cdots \subseteq B_t = B
\]
is a composition series for \(B\), and this gives what we want.

10. We first show that \(W_n\) is finitely generated by induction on \(n\). If \(n = 1\), then we can regard \(W_1\) as a vector space over \(R/\mathfrak{m}\). Now, since \(M\) is an Artinian \(R\)-module, \(W_1\) is an Artinian \(R\)-module, and hence an Artinian vector space over \(R/\mathfrak{m}\). In other words, \(W_1\) is finite dimensional, and therefore finitely generated as an \(R/\mathfrak{m}\)-module and therefore, also as an \(R\)-module.

Now suppose \(n > 1\) and \(W_{n-1}\) is a finitely generated \(R\)-module. Let \(x_1, \ldots, x_n\) be a generating set for \(\mathfrak{m}\) and let \(C\) be the direct sum of \(n\) copies of \(W_{n-1}\). Then the map \(\phi : W_n \rightarrow C\) defined by \(\phi(w) = (x_1w, \ldots, x_nw)\), for all \(w \in W_n\), is an \(R\)-module homomorphism. Since \(W_{n-1}\) is finitely generated over \(R\), \(C\) is finitely generated over \(R\) and therefore \(\text{image}(\phi)\) is finitely generated over \(R\) (as \(R\) is Noetherian). On the other hand, \(W_1\) is the kernel of \(\phi\), so we have an exact sequence \(0 \rightarrow W_1 \rightarrow W_n \rightarrow \text{image}(\phi) \rightarrow 0\), which implies that \(W_n\) is finitely generated.

Finally, let \(x \in M\). Then \(\mathfrak{m}x \supseteq \mathfrak{m}^2x \supseteq \cdots\) is a descending chain of submodules. Suppose the chain stabilizes at the \(n\)th term. Then \(\mathfrak{m}^nx = \mathfrak{m}^{n+1}x = \mathfrak{m}(\mathfrak{m}^nx)\). Since \(\mathfrak{m}^nx\) is finitely generated, Nakayama’s lemma implies that \(\mathfrak{m}^nx = 0\), i.e., \(x \in W_n\). This gives what we want.