Math 915 Homework 4: Due April 18

Throughout $R$ will denote an associative ring with identity. Unless mentioned otherwise, all $R$-modules will be left $R$-modules. As before, work in teams on this homework set. You may consult with me and you may also freely use results established or assumed in class. Please do not consult with students not on your team or not in our class, professors, texts, online material or any other sources.

Let $A$ and $B$ be $R$-modules and $\mathcal{E}(A, B)$ denote the set of equivalence classes of extensions of $A$ by $B$, together with the Baer sum, as defined in problems 4-6 of HW set 3. In problems 1-6, we outline the argument that $\mathcal{E}(A, B)$ is isomorphic to $\text{Ext}_R^1(A, B)$ as abelian groups. Thus, $[\epsilon] \in \mathcal{E}(A, B)$ represents the equivalence class of the extension $\epsilon$ of $A$ by $B$.

1. Recall that $[\overline{0}] \in \mathcal{E}(A, B)$ is the equivalence class of split extensions, in particular, the class of the trivial extension $0 \to B \overset{i}{\to} A \oplus B \overset{\pi}{\to} A \to 0$. Show that $[\overline{0}]$ is the additive identity under Baer sum.

To see that $\mathcal{E}(A, B)$ is an abelian group that is isomorphic to $\text{Ext}_R^1(A, B)$ we will define a correspondence $\Theta : \mathcal{E}(A, B) \to \text{Ext}_R^1(A, B)$ that is bijective and additive. This will simultaneously show that $\mathcal{E}(A, B)$ is an abelian group and $\mathcal{E}(A, B)$ is isomorphic to $\text{Ext}_R^1(A, B)$. In order to define $\Theta$, we fix once and for all a presentation $0 \to K \overset{v}{\to} P \overset{u}{\to} A \to 0$, with $P$ a projective $R$-module. Given an extension $\epsilon : 0 \to B \overset{i}{\to} X \overset{\pi}{\to} A \to 0$, the Comparison theorem for projective $R$-modules (actually, the first step in the proof of the theorem) yields a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & K \\
& \downarrow{\beta} & \downarrow{\alpha} \\
0 & \longrightarrow & B \overset{i}{\longrightarrow} X \overset{\pi}{\longrightarrow} A \longrightarrow 0.
\end{array}
$$

If we apply $\text{Hom}_R(-, B)$ to the presentation sequence, we get the exact sequence

$$(\ast) \quad 0 \to \text{Hom}_R(A, B) \to \text{Hom}_R(P, B) \overset{u^*}{\longrightarrow} \text{Hom}_R(K, P) \overset{i}{\longrightarrow} \text{Ext}_R^1(A, B) \to 0.$$ 

We set $\Theta([\epsilon]) := f(\beta)$.

2. Prove that $\Theta$ is well defined. Hint: First argue that it is enough to prove that we get the same value for $\Theta([\epsilon])$ if we choose a different value for $\alpha$.

3. Prove that $\Theta$ is additive. In other words, let $\epsilon$ as above be an extension of $A$ by $B$, with the associated commutative diagram involving $\epsilon$ and the presentation sequence, together with the exact sequence ($\ast$). Given another extension $\epsilon'$, use the same presentation sequence for $A$, and for the data newly associated to $\epsilon'$, put primes everywhere, so $\Theta([\epsilon']) = f(\beta')$. Thus, one must show
\( \Theta(\epsilon + \epsilon') = f(\beta + \beta') \), since the latter equals \( f(\beta) + f(\beta') \). Hint: For this, first prove that there exists a commutative diagram (\( \star \star \))

\[
\begin{array}{ccc}
0 & \longrightarrow & K \\
\downarrow & & \downarrow \nu \\
B & \longrightarrow & A \\
0 & \longrightarrow & 0,
\end{array}
\]

where the bottom row is the extension defining \( \epsilon + \epsilon' \), the Baer sum of \( \epsilon \) and \( \epsilon' \) (as in problem 6 from HW set 3).

To show that \( \Theta \) is bijective, we will need the push forward coming from the map \( \nu \) in the presentation sequence of \( A \) and an arbitrary homomorphism \( \gamma : K \rightarrow B \). In other words, the pushforward determined by \( \nu \) and \( \gamma \) is the module \((P \oplus B)/W\), where \( W := \{(\nu(k), -\gamma(k)) \mid k \in K\}\).

4. For \( \nu, \gamma \) and \((P \oplus B)/W\) as in the preceding paragraph, show there exists a commutative diagram of exact sequences

\[
\begin{array}{ccc}
0 & \longrightarrow & K \\
\downarrow \gamma & & \downarrow \nu \\
B & \longrightarrow & A \\
0 & \longrightarrow & 0
\end{array}
\]

with well defined homomorphisms \( e : B \rightarrow (P \oplus B)/W \) and \( d : (P \oplus B)/W \rightarrow A \).

5. Define \( \Phi : \text{Ext}^1_R(A, B) \rightarrow \mathcal{E}(A, B) \) as follows. Given \( x^* \in \text{Ext}^1_R(A, B) \), from (\( \star \)) there exists \( \gamma \in \text{Hom}_R(K, P) \) such that \( f(\gamma) = x^* \). We set \( \Phi(x^*) := [\delta] \), where \( \delta \) is the extension obtained in the second row of the diagram in problem 4, i.e., the extension arising from the push forward determined by \( \gamma \). Prove that \( \Phi \) is well defined.

6. Prove either that \( \Theta \circ \Phi \) is the identity on \( \text{Ext}^1_R(A, B) \) or \( \Phi \circ \Theta \) is the identity on \( \mathcal{E}(A, B) \). The proofs are similar. Thus, \( \Theta \) is a bijective abelian group homomorphism, which concludes the proof that \( \text{Ext}^1_R(A, B) \) and \( \mathcal{E}(A, B) \) are isomorphic as abelian groups.

For the rest of this problem set we assume that \( R \) is a commutative, Noetherian ring.

7. Let \( M \) be a finitely generated \( R \)-module. The annihilator of \( M \), denoted \( \text{ann}(M) \), is the set of elements \( \{r \in R \mid rx = 0, \text{ for all } x \in M\} \). Prove that for any prime ideal \( P \subseteq R \), \( M_P \neq 0 \) if and only if \( \text{ann}(M) \subseteq P \). The primes with this property are called the support of \( M \).

8. Let \( M \) be a finitely generated \( R \)-module and \( I \subseteq R \) an ideal. Prove that \( \text{grade}(I, M) \) equals \( \min\{\text{depth}(M_P) \mid I + \text{ann}(M) \subseteq P\} \). In particular, \( \text{grade}(I) \) is the minimum value of \( \text{depth}(R_P) \) among prime ideals \( P \) containing \( I \).

9. For a finitely generated \( R \)-module, the grade of \( M \) is defined to be the grade of the annihilator of \( M \). Prove that \( \text{grade}(M) \leq \text{proj.dim}_R(M) \).

10. Let \( M \) be a finitely generated \( R \)-module such that \( \text{proj.dim}_R(M) < \infty \). \( M \) is said to be a perfect \( R \)-module, if \( \text{grade}(M) = \text{proj.dim}_R(M) \). By convention, if \( M = I \) is an ideal of \( R \), then \( I \) is a perfect ideal if \( R/I \) is a perfect \( R \)-module. Prove that a perfect \( R \)-module \( M \) is grade unmixed, i.e., \( \text{grade}(P) = \text{grade}(M) \), for all \( P \in \text{Ass}(M) \).
11. Let $M$ be a finitely generated $R$-module.

(a) Suppose $(R, \mathfrak{m})$ is local, $\mathfrak{m}^r \subseteq \text{ann}(M)$, for some $r \geq 1$, and $\text{proj.dim}_R(M) < \infty$. Prove that $M$ is a perfect $R$-module. In particular, every $\mathfrak{m}$-primary ideal with finite projective dimension is a perfect ideal.

(b) Suppose $M$ is a perfect $R$-module, and

\[ \mathcal{F} : \quad 0 \to F_n \xrightarrow{\phi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to M \to 0 \]

is a finite free resolution of $M$ with $n = \text{proj.dim}_R(M)$. Prove that the dual resolution

\[ \mathcal{F}^* : \quad 0 \to F_0 \xrightarrow{\phi_0^t} F_1 \to \cdots \to F_{n-1} \xrightarrow{\phi_{n-1}^t} F_n \to C \to 0 \]

is a finite free resolution of $C$, the cokernel of $\phi_n^t$. Here we are writing $\phi_i^t$ to denote the transpose of the matrix $\phi_i$.

12. Upon deleting $M$, assume that $\mathcal{F}$ in the previous problem is just a complex of finitely generated free $R$-modules. Prove that the complexes $\mathcal{F}^*$ and $\text{Hom}_R(\mathcal{F}, R)$ are isomorphic as complexes.

13. Let $(R, \mathfrak{m})$ be a local ring and $M$ a finitely generated, perfect $R$-module with $\text{proj.dim}_R(M) = n$. Prove that $\text{Ext}^n_R(\text{Ext}^n_R(M, R), R)$ is isomorphic to $M$.

14. For an $R$-module $A$, let $E(A)$ denote the injective hull of $A$. For a family of $R$-modules $\{A_\beta\}_{\beta \in B}$, prove that $E(\bigoplus_{\beta \in B} A_\beta) = \bigoplus_{\beta \in B} E(A_\beta)$.

15. Let $A$ be an $R$-module and $S \subseteq R$ a multiplicatively closed set. Prove that $E_R(A)_S$ is isomorphic to $E_{R_S}(A_S)$ as $R_S$-module. Use this to prove that if

\[ 0 \to A \to Q_0 \to Q_1 \to Q_2 \to \cdots \]

is a minimal injective resolution of $A$, then

\[ 0 \to A_S \to (Q_0)_S \to (Q_1)_S \to (Q_2)_S \to \cdots \]

is a minimal injective resolution of $A_S$ over $R_S$. Note, among other things, you will need to show that each $(Q_i)_S$ is an injective $R_S$-module.