UNIQUE FACTORIZATION IN REGULAR LOCAL RINGS

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In this note we prove that every regular local ring of dimension 3 is a unique factorization domain. Nagata* showed (Proposition 11) that if every regular local ring of dimension 3 is a unique factorization domain, then every regular local ring has unique factorization.* Thus, combining these results we have that every regular local ring is a unique factorization domain.

Throughout this note R is a local ring with maximal ideal M. The definitions and notation follow those of Auslander and Buchsbaum.¹

PROPOSITION 1. Let x be in M and y in R such that a = (x):y satisfies the following conditions: (a) hd a ≤ 1 and (b) x is not in Ma. Then a = (x) and x is not a zero divisor.

Proof: Suppose x does not generate a. Since x is not in Ma, there exist a₁,...,aₙ in a (n > 0) such that x, a₁,...,aₙ form a minimal generating set for a. Let Rⁿ⁺¹ denote the direct sum of n + 1 copies of R, define f: Rⁿ⁺¹ → a by f(a₁,...,aₙ) = rox + \sum rᵢaᵢ and let K = Ker f. Since x, a₁,...,aₙ is a minimal generating system for a, we have that f is an epimorphism and K is contained in MIRⁿ⁺¹. From the exact sequence 0 → K → Rⁿ⁺¹ → a → 0 and the fact that hd a ≤ 1, we have that K is R-free. Since a is not principal, we have that hd a = 1 and thus K ≠ 0.

Let Nᵢ = (tᵢ₁,...,tᵢₘ) be a free basis for K over R (i = 1,...,m). Since a₁ is in a, we have that ya₁ = −ux for some v in v in R. Let V = (v, y, 0, ..., 0) and T = (a₁, −x, 0, ..., 0). Then V and T are in K. Let V = \sum rᵢNᵢ and T = \sum sᵢNᵢ. Now xV = −yT. Therefore we have that \sum i=1^m xᵢNᵢ = \sum j=1^m sᵢNᵢ. Since the {Nᵢ} are a free basis for K over R, we have that xᵢ = −sᵢ for all i = 1,...,m. But (x):y = a. Therefore we have that each sᵢ is in a. Since T = (a₁, −x, 0, ..., 0) = \sum j=1^m sᵢNᵢ, it follows that −x = \sum j=1^m sᵢNᵢ. Therefore x is in Ma (since K is contained in MIRⁿ⁺¹) which contradicts the fact that x is not in Ma. Thus a = (x). The fact that x is not a zero divisor follows from the assumption hd a ≤ 1 < ∞ and [2, corollary 6.3].

COROLLARY 2. Suppose p is a prime ideal in R such that dim Rₚ = 1 and hd R/p ≤ 2. Then p is a principal ideal.

Proof: Since hdₚR/p ≥ hdₚRₚ/pRₚ = gl. dim Rₚ, it follows that the gl. dim Rₚ is finite. (See reference 1, 1.6 and reference 3; VIII, 2.6.) Therefore Rₚ is a regular local ring of dimension 1 (see reference 1, 1.9). Let x₁,...,xᵣ be a minimal generating set for p. Then the xᵣ, i = 1,...,r considered as elements in Rₚ generate pRₚ. Since Rₚ is a regular local ring of dimension 1, we have that pRₚ = xᵣRₚ for some j. Let (xᵣ) = q₁ ∧ qₑ. Then qᵣ is a normal, primary decomposition for (xᵣ). From xᵣRₚ = pRₚ, it follows that one of the qᵣ is p. Let us say qᵣ = p. Then for
y in \((q_i \cap \ldots \cap q_1) - p\) we have that \((x_j): y = p\). Since \(x_j\) is not in \(\mathfrak{M} p\) and \(hd p \leq 1\), it follows from the previous proposition that \(p = (x_i)\).

**Theorem 3.** Let \(R\) be a local domain of dimension \(\leq 3\) such that \(hd R/p < \infty\) for all minimal prime ideals \(p\). Then \(R\) is a unique factorization domain.

**Proof:** Since \(R\) is a noetherian domain, it follows from reference 4; Lemma 1, pg. 408, that it suffices to show that each minimal prime ideal is principal in order to show that \(R\) is a unique factorization domain. But by Corollary 2, it will follow that a minimal prime ideal \(p\) is principal if we can show that \(hd R/p \leq 2\). Since \(hd R/p < \infty\) we have by reference 1; 3.7 and 1.3 that \(hd R/p + \text{Codim} R/p = \text{Codim} R \leq \dim R\). But \(\text{Codim} R/p \geq 1\) and \(\dim R \leq 3\). Thus \(hd R/p \leq 2\), which completes the proof.

Since every module has finite homological dimension over a regular local ring, we have established

**Corollary 4.** Every regular local ring of dimension \(\leq 3\) is a unique factorization domain.

**Theorem 5.** Every regular local ring is a unique factorization domain.

Prior to this result, Zariski proved that if every complete regular local ring of dimension 3 is a unique factorization domain, then every complete regular local ring is a unique factorization domain (unpublished). Combining this with Mori's and Krull's result that a local ring is a unique factorization domain if it's completion is a unique factorization domain, we obtain another proof of this reduction theorem.


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**ON THE FALSITY OF EULER'S CONJECTURE ABOUT THE NON-EXISTENCE OF TWO ORTHOGONAL LATIN SQUARES OF ORDER \(4t + 2\)**

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1. **Introduction.**—The purpose of this paper is to prove a general theorem on the existence of pairwise orthogonal Latin squares (p.o.l.s.) of a given order and to give a counter example to Euler's conjecture\(^3\) that there do not exist two p.o.l.s. of order \(4t + 2\).

2. **Definitions.**—An arrangement of \(v\) objects (called treatments) in \(b\) sets (called blocks) will be called a pairwise balanced design of index unity and type \((v; k_1, k_2, \ldots, k_m)\) if each block contains either \(k_1, k_2, \ldots, k_m\) treatments which are all distinct \((k_i \leq v, k_i \neq k_j)\), and every pair of distinct treatments occurs exactly in one block of the design. If the number of blocks containing \(k_t\) treatments is \(b_t\) then clearly