# A Little Math

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# 1 Set Theory

Set theory is the basis of all Mathematics, the place at which assumptions must be made for us to build upon. These assumptions are called axioms; they are never proven, but all theorems must be rigorously provable from these assumptions.

**Axiom 1.1.** (*Axiom of the Empty Set*) There is a set called the empty set denoted $\varnothing$. This set has no elements.

**Axiom 1.2.** (*Axiom of Extension*) If every element of $X$ is an element of $Y$ and every element of $Y$ is an element of $X$ then $X$ and $Y$ are the same set.

**Definition 1.** If every element of $X$ is an element of $Y$ we say $X$ is a **subset** of $Y$ and write $X \subseteq Y$.

**Axiom 1.3.** (*Axiom of Pairing*) Given two sets, $A$ and $B$, there is another set $C = \{A, B\}$ which contains $A$ and $B$ as elements.

**Axiom 1.4.** (*Axiom of Unions*) Given a collection of sets there is a set which has as its elements the elements of each set in the collection.

**Axiom 1.5.** (*Axiom of Infinity*) There is a set containing the number 0 and the successor of each of its elements. This set is called the set of whole numbers.

**Axiom 1.6.** (*Axiom of Specification*) Let $A$ be a set and $P(x)$ be an open sentence for which $P(a)$ may or may not be true where $a$ is an element of $A$. Then those $a$ in $A$ for which $P(a)$ is true form a set.

**Axiom 1.7.** (*Axiom of Powers*) If $A$ is a set then the collection of all subsets of $A$ is a set. This set is called the **power set** of A and denoted $\mathcal{P}(A)$.

**Axiom 1.8.** (*Axiom of Replacement*) Let $P(x, y)$ be any formula of LAST such that for each set $a$ there is a unique set $b$ for which $P(a, b)$ is true. Let $I$ be a set. Then there is a set $S = \{b : P(a, b) \text{ is true for some } a \in I\}$.

**Axiom 1.9.** (*Axiom of Choice*) Given a collection of nonempty sets there is a set which contains exactly one element from each set in the collection.

**Definition 1.10.** Given two sets, $A$ and $B$, the **product of $A$ and $B$**, denoted $A \times B$, is $\{(a, b) : a \in A, b \in B\}$.

*Remark(s)* 1.11. Likewise, given $A_1, A_2, \ldots, A_n$ we can form the product of the $A_i$, denoted $\prod_{i=1}^{n} A_i = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i\}$. Given $\{A_\alpha : \alpha \in \Lambda\}$, the product of the $A_\alpha$ may be thought of as $\{f : \Lambda \to \cup A_\alpha \mid f(\alpha) \in A_\alpha\}$ or as $\{(a_\alpha)_\Lambda : a_\alpha \in A_\alpha\}$ (the set of all $\Lambda$-tuples with the $\alpha^{th}$ entry coming from $A_\alpha$). This product is denoted $\prod_\Lambda A_\alpha$.

**Definition 1.12.** If $S$ is a set then a function, $* : S \times S \to S$, is **binary operation**.

**Definition 1.13.** If $S$ is a set then $R \subseteq S \times S$ is an **equivalence relation** on $S$ iff

1. $R$ is reflexive: $x \in S \implies (x, x) \in R$,
2. symmetric: $(x, y) \in R \implies (y, x) \in R$,
3. and transitive: $(x, y), (y, z) \in R \implies (x, z) \in R$

Usually we will write $xRy$ instead of $(x, y) \in R$.

# 2 Algebra

## 2.1 Abstract Algebra

Abstract algebra is quite different from the so called college algebra. College algebra could be considered a subset of abstract algebra, in that college algebra is the study of the field of real numbers. Abstract algebra, on the other hand, is the study of algebraic structures: groups, rings, fields, etc.

**Definition 2.1.** A **group** is an ordered pair $(S, *)$ where $S$ is a set and $*$ is a binary operation on $S$ satisfying:

1. if $a, b \in S$ then $a * b \in S$,
2. if $a, b, c \in S$ then $a * (b * c) = (a * b) * c$,
3. $\exists e \in S$ s.t. $e * a = a * e \; \forall a \in S$,
4. and $\exists a^{-1} \in S$ s.t. $a * a^{-1} = a^{-1} * a = e \; \forall a \in S$

*Remark(s)* 2.2. From now on, unless necessary, the group operation will be denoted by juxtaposition. That is $a * b$ will be written $ab$. Also rather than saying $(G, *)$ is a group we will say $G$ is a group and the operation should be understood.

**Theorem 2.3.** *If $G$ is a group, then $e$, the identity element, is unique.*

**Theorem 2.4.** *A set $G$ is a group iff the equation $ax = b$ has a unique solution for all $a, b \in G$.*

**Definition 2.5.** A group, $G$, is **abelian** iff the operation on $G$ is commutative, that is, for all $x, y \in G$, $xy = yx$.

**Definition 2.6.** If $R$ is a set then $(R, +, *)$ is a **ring** iff $(R, +)$ is an abelian group, and $*$ is a binary operation on $R$ satisfing

1. $(x * y) * z = x * (y * z)$
2. $x * (y + z) = x * y + x * z$
3. and $(x + y) * z = x * z + y * z \forall x, y, z \in R$.

*Remark(s)* 2.7.

1. Usually we will denote $x * y$ by $xy$ in a ring.
2. The operation $+$ is usually called addition

**Definition 2.8.** A **ring with identity** is a ring in which the multiplication operation has an identity element.

**Definition 2.9.** Field

## 2.2 Linear Algebra

**Definition 2.10.** Vector Space

# 3 Analysis

Analysis is like Calculus.

## 3.1 Topology

Topology is analysis on spaces more general than $\mathbb{R}$ or $\mathbb{C}$.

**Definition 3.1.** A topological space is an ordered pair, $(X, \tau)$, where $X$ is a set and $\tau \subseteq \mathcal{P}(X)$ satisfing:

1. $\{U_\alpha : \alpha \in \Lambda\} \subseteq \tau \implies \bigcup_\Lambda U_\alpha \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. $\varnothing, X \in \tau$

**Definition 3.2.** A function $f : X \to Y$ is **continuous at $\boldsymbol{x}$** iff given $f(x) \in V \in \tau(Y)$ there exists $U \in \tau(X)$ such that $x \in U$ and $f[U] \subseteq V$. A function is continuous on $X$ if it is continuous at each point in $X$.

# 4 Numerical Analysis

1. a

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