

ON INJECTIVITY OF MAPS BETWEEN GROTHENDIECK GROUPS INDUCED BY COMPLETION

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ABSTRACT. We give an example of a local normal domain R such that the map of Grothendieck groups $G(R) \rightarrow G(\hat{R})$ is not injective. We also raise some questions about the kernel of that map.

1. INTRODUCTION

Let (R, m, k) be a local ring and \hat{R} the m -adic completion of R . Let $\mathcal{M}(R)$ be the category of finitely generated R -modules. The Grothendieck group of finitely generated modules over R is defined as:

$$G(R) = \frac{\bigoplus_{M \in \mathcal{M}(R)} \mathbb{Z}[M]}{\langle [M_2] - [M_1] - [M_3] \mid 0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0 \text{ is exact} \rangle}$$

In [KK], Kanoi and Kurano studied injectivity of the map $G(R) \rightarrow G(\hat{R})$ induced by flat base-change. They showed that such map is the injective in the following cases : 1) R is Henselian, 2) R is the localization at the irrelevant ideal of a positively graded ring over a field, or, 3) R has only isolated singularity. Their results raise the question: Is the map between Grothendieck group induced by completion always injective?

In [Ho1], Hochster announced a counterexample to the above question:

Theorem 1.1. *Let k be a field. Let $R = k[x_1, x_2, y_1, y_2]_{(x_1, x_2, y_1, y_2)} / (x_1x_2 - y_1x_1^2 - y_2x_2^2)$. Let $P = (x_1, x_2)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $G(R) \rightarrow G(\hat{R})$. However $[M] \neq 0$ in $G(R)$.*

Hochster's example comes from the "direct summand hypersurface" in dimension 2 and is not normal. He predicted that there is also an example which is normal. The main purpose of this note is to provide such an example. We have:

Proposition 1.2. *Let $R = \mathbb{R}[x, y, z, w]_{(x, y, z, w)} / (x^2 + y^2 - (w+1)z^2)$. R is a normal domain. Let $P = (x, y, z)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $G(R) \rightarrow G(\hat{R})$. However $[M] \neq 0$ in $G(R)$.*

This will be proved in Section 2. We note that Kurano and Srinivas has recently constructed an example of a local ring R such that the map $G(R)_{\mathbb{Q}} \rightarrow G(\hat{R})_{\mathbb{Q}}$ is not injective (see [KS]). The ring in their example is not normal, and we do not know if a normal example exists in that context (i.e. with rational coefficients).

In section 3 we will discuss some questions on the kernel of the map $G(R) \rightarrow G(\hat{R})$.

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unpublished note [Ho2], which provided the key ideas for our example. We also thank the referee for many helpful comments.

2. OUR EXAMPLE

We shall prove Proposition 1.2. First we need to recall some classical results:

Corollary 2.1. (*Swan, [Sw], Corollary 11.8*) *Let k be a field of characteristic not 2, $n > 1$ an integer and $R = k[x_1, \dots, x_n]/(f)$ where f is a non-degenerate quadratic form in $k[x_1, \dots, x_n]$. Then $G(R) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if $C_0(f)$, the even part of the Clifford algebra of f , is simple.*

Proposition 2.2. (*Samuel, see [Fo], Proposition 11.5*) *Let k be a field of characteristic not 2 and f be a non-degenerate quadratic form in $k[x_1, x_2, x_3]$. Let $R = k[x_1, x_2, x_3]/(f)$. If $f = 0$ has no non-trivial solution in k then $\text{Cl}(R) = 0$.*

Proposition 2.3. (*Kamoi-Kurano*) *Let $S = \bigoplus_{n \geq 0} S_n$ be a graded ring over a field S_0 and $S_+ = \bigoplus_{n > 0} S_n$. Let $A = S_{S_+}$. Then the map $G(S) \rightarrow G(A)$ induced by localization is an isomorphism.*

Proof. See the proof of Theorem 1.5 (ii) in [KK]. \square

Proposition 1.2 now follows from the following Propositions (clearly, R is normal, since the singular locus $V(x, y, z)$ has codimension 2):

Proposition 2.4. $[\hat{M}] = 0$ in $G(\hat{R})$.

Proof. $\hat{R} = \mathbb{R}[[x, y, z, w]]/(x^2 + y^2 - (w+1)z^2)$. We want to show that $[\hat{R}/P\hat{R}] = 0$ in $G(\hat{R})$. Let $\alpha = \sqrt{w+1}$ which is a unit in \hat{R} . Let $Q = (x, y - \alpha z)\hat{R}$. Then clearly Q is a height 1 prime in \hat{R} and $P\hat{R} = Q + (y + \alpha z)\hat{R}$. The short exact sequence:

$$0 \rightarrow \hat{R}/Q \rightarrow \hat{R}/Q \rightarrow \hat{R}/P\hat{R} \rightarrow 0$$

where the second map is the multiplication by $y + \alpha z$ shows that $[\hat{R}/P\hat{R}] = 0$ in $G(\hat{R})$. \square

Proposition 2.5. $[M] \neq 0$ in $G(R)$.

Proof. It is enough to show that $[M_P] \neq 0$ in $G(R_P)$. Let $K = \mathbb{R}(w)$ then $R_P \cong K[x, y, z]_{(x, y, z)}/(f)$ where $f = x^2 + y^2 - (w+1)z^2$. Let $S = K[x, y, z]/(f)$. Clearly f is a non-degenerate quadratic form. Since the rank of f is 3, an odd number, $C_0(f)$ is a simple algebra over K (see, for example, [La], Chapter 5, Theorem 2.4). By 2.1 and 2.3, $G(R_P) = G(S) = \mathbb{Z} \oplus \mathbb{Z}/(2)$. We claim that f has no non-trivial solution in K . Suppose it has. Then by clearing denominators, there are polynomials $a(w), b(w), c(w) \in \mathbb{R}[w]$ such that

$$a(w)^2 + b(w)^2 = (w+1)c(w)^2.$$

The degree of $a(w)^2 + b(w)^2$ is always even. The degree of $(w+1)c(w)^2$ is odd unless $c(w) = 0$. But then $a(w)^2 + b(w)^2 = 0$ which forces $a(w) = b(w) = 0$, a contradiction. By the claim and 2.2, $\text{Cl}(R_P) = \text{Cl}(S) = 0$. Thus $[R_P]$ and $[R_P/PR_P]$ generate $G(R_P) = \mathbb{Z} \oplus \mathbb{Z}/(2)$ (since the Grothendieck group is generated by $\{[R_P/Q], Q \in \text{Spec } R_P\}$ and $\dim R_P = 2$). Since $\mathbb{Z} \oplus \mathbb{Z}/(2)$ can not be generated by one element, $[R_P/PR_P]$ must be nonzero (it is easy to see that $[R_P/PR_P]$ is 2-torsion). \square

3. ON THE KERNEL OF THE MAP $G(R) \rightarrow G(\hat{R})$

In this section we raise some questions about the kernel of the map $G(R) \rightarrow G(\hat{R})$. First we fix some notations. Throughout this section we will assume, for simplicity, that R is excellent, and is a homomorphic image of a regular local ring T . Let $d = \dim R$. Let $A_i(R)$ be the i -th Chow group of R , i.e.,

$$A_i(R) = \frac{\bigoplus_{P \in \text{Spec } R, \dim R/P=i} \mathbb{Z} \cdot [\text{Spec } R/P]}{\langle \text{div}(Q, x) \mid Q \in \text{Spec } R, \dim R/Q = i+1, x \in R \setminus Q \rangle}$$

where

$$\text{div}(Q, x) = \sum_{P \in \text{Min}_R R/(Q, x)} l_{R_P}(R_P/(Q, x)R_P)[\text{Spec } R/P].$$

For an abelian group A , we let $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. The Chow group of R is defined to be $A_*(R) = \bigoplus_{i=0}^d A_i(R)$. It is well known that there is a \mathbb{Q} -vector space isomorphism:

$$\tau_{R/T} : G(R)_{\mathbb{Q}} \rightarrow A_*(R)_{\mathbb{Q}}$$

(It is unknown whether this is independent of T). We also remark that the Grothendieck group $G(R)$ admits a filtration by $F_i G(R) = \langle [M] \in G(R) \mid \dim M \leq i \rangle$.

The existing examples on the failure of injectivity for the map $G(R) \rightarrow G(\hat{R})$ and the affirmative results in [KK] motivate the following question:

Question 3.1. *Suppose that R satisfies (R_n) (i.e., regular in codimension n). Then is $\ker(G(R) \rightarrow G(\hat{R}))$ contained in $F_{d-n-1} G(R)$?*

Question 3.1 is closely related to the following:

Question 3.2. *Suppose that R satisfies (R_n) . Then is the map $A_i(R) \rightarrow A_i(\hat{R})$ injective for $i \geq d - n$?*

In fact, if we allow rational coefficients, then the previous questions are equivalent. Let $G^i(R) = F_i G(R)/F_{i-1} G(R)$. Then clearly we have a decomposition:

$$G(R)_{\mathbb{Q}} = \bigoplus_{i=0}^d G^i(R)_{\mathbb{Q}}$$

Also, the Riemann-Roch map decomposes into isomorphisms $\tau^i : G^i(R)_{\mathbb{Q}} \rightarrow A_i(R)_{\mathbb{Q}}$, which make the following diagram:

$$\begin{array}{ccc} G^i(R) & \xrightarrow{\tau_{R/T}^i} & A_i(R) \\ \downarrow g_i & & \downarrow f_i \\ G^i(\hat{R}) & \xrightarrow{\tau_{\hat{R}/\hat{T}}^i} & A_i(\hat{R}) \end{array}$$

commutative. It follows that

$$\ker(G(R)_{\mathbb{Q}} \rightarrow G(\hat{R})_{\mathbb{Q}}) \cong \bigoplus_i^d \ker(f_i) \cong \bigoplus_i^d \ker(g_i).$$

Thus we have:

Proposition 3.3. *Let R be an excellent local ring which is a homomorphic image of a regular local ring. Let $\dim R = d$ and let $0 < l \leq d$ be an integer. Then the maps $A_i(R)_{\mathbb{Q}} \rightarrow A_i(\hat{R})_{\mathbb{Q}}$ are injective for $i \geq l$ if and only if $\ker(G(R)_{\mathbb{Q}} \rightarrow G(\hat{R})_{\mathbb{Q}}) \subseteq F_{l-1}G(R)_{\mathbb{Q}}$.*

We do not know if 3.2 is true even if $l = 1$. Note that if R is normal, then both 3.1 and 3.2 are true for $l = 1$. In that situation $A_1(R) \cong \text{Cl}(R)$, and the map between class groups of R and \hat{R} is injective. Furthermore, it is well known that $G(R)/F_{d-2}G(R) \cong A_d(R) \oplus A_{d-1}(R)$ (see, for example [Ch], Corollary 1), so 3.1 is also true for $l = 1$.

Finally, one could formulate a stronger version of 3.1 as follows. Note that in both Hochster's example and the example presented here, the support of the modules given actually equal to the singular locus of R . So one could ask:

Question 3.4. *Let R be an excellent local ring. Let $X = \text{Spec } R$, $Y = \text{Sing}(X)$, $\hat{X} = \text{Spec } \hat{R}$ and $\hat{Y} = \text{Sing}(\hat{X})$. One then has a commutative diagram:*

$$\begin{array}{ccc} G(Y) & \xrightarrow{f} & G(X) \\ \downarrow & & \downarrow g \\ G(\hat{Y}) & \longrightarrow & G(\hat{X}) \end{array}$$

(Here $G(X)$ denotes the Grothendieck group of coherent \mathcal{O}_X -modules and the maps are naturally induced by closed immersions or flat morphisms). Is $\ker(g)$ contained in $\text{im}(f)$?

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