Math 791 Spring 2014 Final

Q1. Find the number of solutions to the equation $x^2 = 1$ in the ring $\mathbb{Z}_n$ for each integer n.

Q2. Show that the polynomial $x^{n-1} + x^{n-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$ if and only if $n = p$ is a prime. (for the direction when $n = p$, make a change of variable $x \to x + 1$ and use Eisenstein criterion).

Extra Credit: If f is a irreducible integral polynomial in $\mathbb{Q}[x]$, can f be changed, with a change of variable, to something satisfying Eisenstein criterion?

Q3. For any prime p find the number of monic irreducible polynomials of degree 2 over $\mathbb{Z}_p$. Do the same problem for degree 3.

Extra Credit: Generalize the above statement to higher degree polynomials as much as you can.

Q4. Show that the quotient field (section 4.7) of $\mathbb{Z}[\sqrt{2}]$ is $\mathbb{Q}[\sqrt{2}]$.

Extra Credit: Generalize the above statement as much as you can.

Q5. Find the number of subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ (p is a prime number). How about the number of subgroups up to isomorphism?

Extra Credit: Generalize and solve the above problem as much as you can.

Q6. Prove that an Eucledian domain (section 4.5) is a principal ideal domain.

Extra Credit: Let k be your favorite field. Classify all polynomials $f(x, y) \in k[x, y]$ such that $R = k[x.y]/(f)$ is an Eucledian domain.

Q7. Let k be a positive integer. In how many way one can color the edges of an equilateral triangle using k colors (two coloring schemes are considered the same if one can be obtained from the other via some symmetry of the triangle)?

Extra Credit: Describe in details an application of group actions on a set that you enjoy (does not have to come from class).