An improper integral is a definite integral (A) over infinite intervals (infinite domain), or (B) with an integrand which has infinite discontinuity.

When integrating with unbounded limits of integration, rewrite the improper integral as the limit of proper integrals.

\[ \int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx. \]

If \( f \) has an infinite discontinuity at \( x = c \) where \( c \) lies in \([a,b]\), then

\[ \int_a^b f(x)\,dx = \left( \lim_{t \to c^-} \int_a^c f(x)\,dx \right) + \left( \lim_{t \to c^+} \int_c^b f(x)\,dx \right). \]

**An important example:** \( \int_1^\infty \frac{1}{x^p}\,dx \) converges when \( p > 1 \) and diverges when \( p \leq 1 \).

\( \int_0^1 \frac{1}{x^p}\,dx \) converges when \( p < 1 \) and diverges when \( p \geq 1 \).

**Comparison Theorem** Suppose that \( f \) and \( g \) are continuous functions with \( f(x) \geq g(x) \geq 0 \) for \( x \geq a \). (a) If \( \int_a^\infty f(x)\,dx \) is convergent, then \( \int_a^\infty g(x)\,dx \) is convergent. (b) If \( \int_a^\infty g(x)\,dx \) is divergent, then \( \int_a^\infty f(x)\,dx \) is divergent.

**Practice Problems**

1. Which of the following integrals is improper? Explain your answer, but do not evaluate the integral.

   (a) \( \int_0^1 e^{-x}\,dx \)

   (b) \( \int_0^\pi \sec(x)\,dx \)

   (c) \( \int_0^\infty \sin(x)\,dx \)

   (d) \( \int_0^1 \frac{1}{\sqrt{3-x^2}}\,dx \)

   (e) \( \int_0^3 \ln|x|\,dx \)

2. Show that \( \int_1^\infty \frac{1}{x^3 + 4x + 2}\,dx \) converges by comparing it to \( \int_1^\infty x^{-3}\,dx \).

3. Evaluate the improper integrals:

   (a) \( \int_{-4}^0 (x+2)^{-\frac{3}{2}}\,dx \)

   (b) \( \int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \)

   (c) \( \int_0^\infty \sin(x)\,dx \)

   (d) \( \int_0^{\frac{\pi}{2}} \tan(x)\,dx \)

   (e) \( \int_0^1 \ln|x|\,dx \)

   (f) \( \int_{-\infty}^\infty xe^x\,dx \)