

Solutions to Practice sets for Exam II

1. The characteristic equation is

$$r^2 + 4r + 5 = 0,$$

with solutions

$$r_{1,2} = -2 \pm 2i.$$

Hence the general solution is

$$y(t) = e^{-2t}(c_1 \cos(2t) + c_2 \sin(2t)).$$

We identify the constants next using initial values. First,

$$y' = -2e^{-2t}(c_1 \cos(2t) + c_2 \sin(2t)) + e^{-2t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)).$$

Hence

$$\begin{aligned} 1 &= c_1 \\ 0 &= -2c_1 + 2c_2 \end{aligned}$$

That is $c_1 = 1$, $c_2 = 1$,

$$y(t) = e^{-2t}(-\cos(2t) + \sin(2t)).$$

2. The characteristic equation is

$$r^2 - 2r - 3 = (r - 3)(r + 1) = 0$$

which has solutions $r_1 = 2, r_2 = -1$. The general solution for homogenous ode

$$y'' - 2y' - 3y = 0$$

is

$$y(t) = c_1 e^{2t} + c_2 e^{-t}.$$

We want a special solution to the in-homogenous ode next. Take a guess

$$Y(t) = Ate^{2t} + Be^{2t}.$$

Then

$$\begin{aligned} Y' &= 2Ate^{2t} + (A + 2B)e^{2t} \\ Y'' &= 4Ate^{2t} + (4A + 4B)e^{2t} \end{aligned}$$

Therefore

$$Y'' - 2Y' - 3Y = 3te^{2t}$$

implies

$$\begin{aligned} -3A &= 3 \\ -3B &= 0 \end{aligned}$$

Hence the solution for the in-homogenous equation is

$$y(t) = c_1 e^{2t} + c_2 e^{-t} - te^{2t}, \quad -\infty < c_1, c_2 < \infty.$$

3. skip this word problem

The characteristic is

$$2r^2 - 3r + 1 = (2r - 1)(r - 1) = 0$$

which has solutions $r = 1/2, r = 1$.

The general solution is therefore

$$y(t) = c_1 e^{t/2} + c_2 e^t.$$

We identify c_1, c_2 next. Since

$$y' = \frac{c_1}{2} e^{t/2} + c_2 e^t,$$

we conclude

$$\begin{aligned} 2 &= c_1 + c_2 \\ \frac{1}{2} &= \frac{c_1}{2} + c_2. \end{aligned}$$

This gives $c_1 = 3$ and $c_2 = -1$.

The solution is

$$y(t) = 3e^{t/2} - e^t.$$

4. The characteristic equation for the homogenous version of the ODE is

$$r^2 - 6r + 8 = (r - 2)(r - 4) = 0.$$

Therefore the general solution for the homogenous version is

$$y(t) = c_1 e^{2t} + c_2 e^{4t}.$$

To identify a special solution Y for the in-homogenous version, we conjecture

$$Y(t) = At + B + Ce^{2t},$$

and note

$$\begin{aligned} Y' &= A + 2Ce^{2t} \\ Y'' &= 4Ce^{2t}. \end{aligned}$$

Therefore

$$\begin{aligned} -6A + 8B &= 0 \\ 8A &= 1 \\ 4C - 12C + 8C &= 1 \end{aligned}$$

The last equality will never happen ! The form of Y we guessed in the beginning is not good. We will try

$$Y(t) = At + B + Ce^{2t} + Dte^{2t}$$

next. Then

$$\begin{aligned} Y' &= A + (2C + D)e^{2t} + 2Dte^{2t} \\ Y'' &= (4C + 4D)e^{2t} + 4Dte^{2t}. \end{aligned}$$

Therefore, plug in the equation,

$$\begin{aligned} -6A + 8B &= 0 \\ 8A &= 1 \\ 4C + 4D - 12C - 6D + 8C &= 1 \\ 4D - 12D + 8D &= 0 \end{aligned}$$

which gives

$$A = \frac{1}{8}, B = \frac{3}{32}, D = -\frac{1}{2}.$$

The general solution to the in-homogenous equation is

$$y(t) = c_1 e^{2t} + c_2 e^{4t} + \frac{1}{8}t + \frac{3}{32} - \frac{1}{2}te^{2t}.$$

5. Skip the word problem.
6. you may want to review "variation of constant method for second order ODEs" which is not included in this practice set.