Study guide for the Math 115 midterm
Fall 2012

This study guide is designed to help you learn the material covered on the Math 115 midterm. Problems on the midterm may differ significantly from these problems in their details, but they will not include material not covered by these problems. If you can do these problems, you should do well on the midterm.

There are 52 review problems. Be sure to print out all of them. And, a bit of reassurance: the actual midterm will have many fewer problems.

Part I: calculations and theory

1. \( \lim_{x \to 1} \frac{x^2 + 5x + 6}{x + 1} = \)
   (a) -6
   (b) -5
   (c) -1
   (d) 0
   (e) 1
   (f) 5
   (g) 6
   (h) the limit does not exist

2. \( \lim_{x \to 1} \frac{x^2 + 5x + 6}{x - 1} = \)
   (a) -6
   (b) -5
   (c) -1
   (d) 0
   (e) 1
   (f) 5
   (g) 6
   (h) the limit does not exist
3. \[ \lim_{x \to -3} \frac{x^2 + 5x + 6}{x + 3} = \]
(a) -6
(b) -5
(c) -1
(d) 0
(e) 1
(f) 5
(g) 6
(h) the limit does not exist

4. \[ \lim_{x \to \infty} \frac{x^2 + 5x + 6}{x + 3} = \]
(a) -6
(b) -5
(c) -1
(d) 0
(e) 1
(f) 5
(g) 6
(h) the limit does not exist

5. \[ \lim_{x \to \infty} \frac{x + 3}{x^2 + 5x + 6} = \]
(a) -6
(b) -5
(c) -1
(d) 0
(e) 1
(f) 5
(g) 6
(h) the limit does not exist
6. \[ \lim_{x \to \infty} \frac{x^2 + 5x + 6}{x^2 + 3} = \]
   (a) -6  
   (b) -5  
   (c) -1  
   (d) 0  
   (e) 1  
   (f) 5  
   (g) 6  
   (h) the limit does not exist

7. If we correctly calculate the derivative of \( f(x) = x^2 + 17 \) directly from the definition of derivative (4-step method), which of the following will appear in our calculations?
   (a) \( \lim_{h \to 0} x^2 + h \)
   (b) \( \lim_{x \to 0} x^2 + h \)
   (c) \( \lim_{x \to h} x^2 + h \)
   (d) \( \lim_{h \to 0} 2x + h \)
   (e) \( \lim_{x \to 0} 2x + h \)
   (f) \( \lim_{x \to h} 2x + h \)

8. Let \( y = x^{17} \). What is the definition of \( \frac{dy}{dx} \)?
   (a) \( \lim_{h \to 0} \frac{x^{17}}{h} \)
   (b) \( \lim_{x \to h} \frac{x^{17}}{h} \)
   (c) \( \lim_{x \to 0} \frac{x^{17}}{h} \)
   (d) \( \lim_{h \to 0} \frac{(x + h)^{17}}{h} \)
   (e) \( \lim_{x \to h} \frac{(x + h)^{17}}{h} \)
   (f) \( \lim_{x \to 0} \frac{(x + h)^{17}}{h} \)
   (g) \( \lim_{h \to 0} \frac{(x + h)^{17} - x^{17}}{h} \)
   (h) \( \lim_{x \to h} \frac{(x + h)^{17} - x^{17}}{h} \)
   (i) \( \lim_{x \to 0} \frac{(x + h)^{17} - x^{17}}{h} \)
9. The slope of the tangent line to the graph \( y = ax^{1/3} \) (where \( a \) is a non-zero constant) at the point \( x = 0 \) is:

(a) \( a \)
(b) \( \frac{-2a}{3} \)
(c) \( \frac{a}{3} \)
(d) 0
(e) there is a vertical tangent line, so the slope is undefined
(f) there is no tangent line, so the slope is undefined

10. The equation of the tangent line to the graph of \( y = ax^3 \) (where \( a \) is a constant) at the point \((1, a)\) is:

(a) \( y = ax \)
(b) \( y = ax - 2a \)
(c) \( y = ax - 2a + 2 \)
(d) \( y = 3ax \)
(e) \( y = 3ax - 2a \)
(f) \( y = 3ax - 2a + 2 \)

Problems 11 and 12 are about the function \( g(x) = x^3 - 3a^2x \) where \( a > 0 \).

11. Where is \( g \) increasing?
(a) exactly when \( x > a \)
(b) exactly when \( x < a \)
(c) exactly when \( x > -a \)
(d) exactly when \( x < -a \)
(e) exactly when \( -a < x < a \)
(f) exactly when either \( x < -a \) or \( x > a \)
(g) \( f \) is always increasing
(h) \( f \) is never increasing

12. What is \( g''(a) \)?
(a) 0
(b) \( 6a \)
(b) \( 3a^2 \)
(d) \( 6a^2 \)
(c) \( -2a^3 \)
(d) \( -3a^3 \)
13. Let $C$ be the cost function for a manufacturing process. Which of the following best estimates the difference between the total cost of producing 500 items and the total cost of producing 501 items?

(a) $C(500)$
(b) $C'(500)$
(c) $\frac{C(500)}{500}$
(d) $\frac{C'(500)}{500}$
(e) $\frac{500C'(500) - C(500)}{250,000}$

14. Let $C$ be the cost function for a manufacturing process. Which of the following is the average cost of producing 500 items?

(a) $C'(500)$
(b) $C'(500)$
(c) $\frac{C(500)}{500}$
(d) $\frac{C'(500)}{500}$
(e) $\frac{500C'(500) - C(500)}{250,000}$

15. Let $h(x) = 2x^3 - 3x^2$. Which of the following is not a consequence of the Intermediate Value Theorem?

(a) There is a number $x$ between 2 and 3 so $h(x) = 3$
(b) There is a number $x$ between 2 and 3 so $h(x) = 5$
(c) There is a number $x$ between 2 and 3 so $h(x) = 10$
(d) There is a number $x$ between 2 and 3 so $h(x) = 17$
(e) All of the above are consequences of the Intermediate Value Theorem.

16. If $3x + y^2 - xy = 0$, what is $\frac{dy}{dx}$?

(a) $2y - x$
(b) $y - 3$
(c) $\frac{y - 3}{2y - x}$
(d) $\frac{2y - x}{y - 3}$
17. If you use the differential to approximate $\sqrt[3]{985}$, what answer do you get (to the nearest 10,000th)?\(^1\)

(a) 9.94
(b) 9.9493
(c) 9.9497
(d) 9.95
(e) 9.9503
(f) 9.9507

Questions 18 through 21 are about the following function:

\[
f(x) = \begin{cases} 
  x & \text{if } x < -1 \\
  x^2 & \text{if } -1 \leq x < 2 \\
  x + 2 & \text{if } 2 \leq x \leq 5 \\
  2x & \text{if } 5 < x 
\end{cases}
\]

18. What is $\lim_{x \to -1^-} f(x)$?
   (a) -2
   (b) -1
   (c) 0
   (d) 1
   (e) 2
   (f) the limit does not exist

19. What is $\lim_{x \to -1^+} f(x)$?
   (a) -2
   (b) -1
   (c) 0
   (d) 1
   (e) 2
   (f) the limit does not exist

20. How many points of discontinuity does $f$ have?
   (a) 0
   (b) 1
   (c) 2
   (d) 3

\(^1\)If you use your calculator to calculate the exact answer, your answer will be wrong.
21. Which of the following is true?
   (a) \( f(5) = 5 \)
   (b) \( f(5) = 7 \)
   (c) \( f(5) = 10 \)
   (d) \( f(5) = 25 \)

Questions 22 and 23 are about the function \( g(x) \) with the following graph.

22. In the region shown, which statement is true?
   (a) There is a derivative at every point.
   (b) There is a point where there is a tangent line but no derivative.
   (c) There is a point where there is no tangent line.

23. In the region shown, which of the following statements is true?
   (a) if \( x \neq 0 \) then \( g'(x) < 0 \)
   (b) if \( x \neq 0 \) then \( g'(x) > 0 \)
   (c) if \( x < 0 \) then \( g'(x) < 0 \) and if \( x > 0 \) then \( g'(x) > 0 \)
   (d) if \( x < 0 \) then \( g'(x) > 0 \) and if \( x > 0 \) then \( g'(x) < 0 \)
24. Consider the graph of the function below in the region shown.

Which statement is true?
(a) The function has a derivative at every point.
(b) There is exactly one point at which the function has no derivative.
(c) There are exactly two points at which the function has no derivative.
(d) There are exactly three points at which the function has no derivative.
(e) There are exactly four points at which the function has no derivative.
(f) There are exactly five points at which the function has no derivative.

Questions 25 and 26 are about the graph of the following function \( f(x) \):

25. Which of the following statements is true?
(a) the function is continuous everywhere between \( x = -2 \) and \( x = 2 \)
(b) the function fails to be continuous at exactly one point between \( x = -2 \) and \( x = 2 \)
(c) the function fails to be continuous at exactly two points between \( x = -2 \) and \( x = 2 \)
(d) the function fails to be continuous at exactly three points between \( x = -2 \) and \( x = 2 \)

26. What is \( \lim_{x \to 1^-} f(x) \)?
(a) -4
(b) -1
(c) 0
(d) 1
(e) 4
Questions 27 and 28 are about the following two functions:

27. Which of the following statements is true?
   (a) $g'(2) > f'(2)$
   (b) $g'(2) = f'(2)$
   (c) $g'(2) < f'(2)$

28. Which of the following statements is true?
   (a) in the region shown, $g'(x) < 0$ and $f'(x) < 0$
   (b) in the region shown, $g'(x) < 0 < f'(x)$
   (c) in the region shown, $f'(x) < 0 < g'(x)$
   (d) in the region shown, $f'(x) > 0$ and $g'(x) > 0$

Part II: word problems

Problems 29 through 32 are about the following situation: A raptor flies straight down towards its prey so that at time $t$ (in hundredths of a second) after it began its dive its height $h = \sqrt{25 - t}$ feet.

29. Which of the following is true for $t$?
   (a) $0 \leq t \leq 5$
   (b) $0 \leq t \leq 25$
   (c) $0 \leq t < \infty$
   (d) $5 \leq t < \infty$
   (e) $25 \leq t < \infty$

30. Which of the following is true for $h$?
   (a) $0 \leq h \leq 5$
   (b) $0 \leq h \leq 25$
   (c) $5 \leq h < \infty$
   (d) $25 \leq h < \infty$
31. The raptor hits the ground (i.e., \( h = 0 \)) when
   (a) \( t = 0 \)
   (b) \( t = 5 \)
   (c) \( t = 10 \)
   (d) \( t = 25 \)
   (e) it never hits the ground

32. Wait a minute! The function \( h = \sqrt{25 - t} \) can’t happen in this situation! It can’t happen because
   (a) when \( t = 0 \) the height is undefined
   (b) when \( t = 25 \) the height is undefined
   (c) when \( t = 0 \) the velocity is undefined
   (d) when \( t = 25 \) the velocity is undefined

Problems 33, 34 and 35 are about the following situation: An art gallery has found that the price of an etching is described by the demand function \( p = A - x^2 \) where \( p \) is the price people are willing to pay, \( x \) is the number of etchings produced from the original, and \( A \) is a positive constant depending on the reputation of the artist (i.e., a famous artist will have a high \( A \) value).

33. What is the revenue when \( x \) etchings are made from the same original and all of them are sold?
   (a) \( A - x^2 \)
   (b) \( Ax - x^2 \)
   (c) \( Ax - x^3 \)
   (d) there isn’t enough information to decide

34. What is the marginal revenue when \( x = 10 \)?
   (a) \( A - 300 \)
   (b) \( A - 100 \)
   (c) \( A - 300 \)
   (d) -30
   (e) -100
   (f) -300
   (g) 30
   (h) 100
   (i) 300
35. Use differentials to find the approximate difference in price when the number of etchings produced is 102 compared to 100 (assume all are sold).\(^2\)

   (a) $2 more
   (b) $2 less
   (c) $3 more
   (d) $3 less
   (e) $4 more
   (f) $4 less
   (g) $400 more
   (h) $400 less
   (i) $402 more
   (j) $402 less
   (k) $404 more
   (l) $404 less
   (m) there is not enough information to decide

36. A new toy is catching on gradually, showing a steady growth in sales, so that \( t \) months after being put on the market \( x = \sqrt{t} \) thousand are sold. The price \( p = \sqrt{100 - x^2}, 1 \leq x \). When \( p = 5 \), what’s \( t \)?

   (a) 0
   (b) 10
   (c) 25
   (d) 50
   (e) 75
   (f) 90
   (g) 100

37. How fast is it going when \( t = 3 \)?

   (a) 1 mile a minute
   (b) 2 miles a minute
   (c) 3 miles a minute
   (d) 6 miles a minute
   (e) 8 miles a minute
   (f) 9 miles a minute

Problems 37 and 38 are about the following situation: A truck’s gas pedal is stuck and the truck is barreling down the highway so that at time \( t \) its distance from the starting point is \( t^2 \) miles, where \( t \) is in minutes.

37. How fast is it going when \( t = 3 \)?

   (a) 1 mile a minute
   (b) 2 miles a minute
   (c) 3 miles a minute
   (d) 6 miles a minute
   (e) 8 miles a minute
   (f) 9 miles a minute

\(^2\)As always, the exact answer will be marked incorrect.
38. When is its acceleration equal to 3 mph²?
   (a) when \( t = 10 \)
   (b) when \( t = 5 \)
   (c) when \( t = 0 \)
   (d) its acceleration never equals 3 mph²!

**Problems 39 through 42** are about the following situation: Michael and Angela have a business selling postcards with scenes of Kansas prairies. Their camera, computer, printer, and paper cutter cost them $2500. Their supplies (e.g., paper, ink cartridges, gas for the car) cost them $0.50 per postcard. Each postcard sells for $1. Since they sell over the internet, they make exactly as many as they sell. Let \( x \) designate the number of postcards they sell.

39. What is their cost in dollars?
   (a) 2500
   (b) \( \frac{x}{2} \)
   (c) \( 2500 + \frac{x}{2} \)
   (d) \( \frac{2500}{x} \)
   (e) \( \frac{2500 + x}{2} \)
   (f) \( x \)
   (g) \( x - \frac{x}{2} \)
   (h) \( x - 2500 - \frac{x}{2} \)
   (i) \( x - \frac{2500}{x} \)
   (j) \( x - \frac{2500 + x}{2} \)
40. What is their profit?
   (a) 2500
   (b) \( \frac{x}{2} \)
   (c) 2500 + \( \frac{x}{2} \)
   (d) \( \frac{2500}{x} \)
   (e) \( \frac{2500 + \frac{x}{2}}{x} \)
   (f) \( x \)
   (g) \( x - \frac{x}{2} \)
   (h) \( \frac{x}{2} - 2500 \)
   (i) \( x - \frac{2500}{x} \)
   (j) \( x - \frac{2500 + \frac{x}{2}}{x} \)

41. What is their marginal profit?
   (a) 0
   (b) $0.10
   (c) $0.25
   (d) $0.50
   (e) $0.75
   (f) $1
   (g) it varies according to how much they sell

42. How many postcards must they sell to break even?
   (a) 5000
   (b) 12,500
   (c) 25,000
   (d) 50,000
   (e) 75,000
   (f) 100,000
Problems 43 through 46 are about the following situation: Wiley E. Coyote, in his eagerness to catch the Roadrunner, jumps straight up in the air from the edge of a cliff so that when he comes down he falls off the cliff. The cliff is 40 feet off the ground. His height \( t \) seconds after he jumps is 

\[ s = -16t^2 + 12t + 40. \]

43. When does he hit the ground? [Hint: \(-16t^2 + 12t + 40 = (4t - 8)(-4t - 5)\).]
   (a) when \( t = 0 \)
   (b) when \( t = 5/4 \)
   (c) when \( t = 2 \)
   (d) when \( t = 4 \)
   (e) when \( t = 5 \)
   (f) when \( t = 8 \)

44. What’s his velocity when \( t = 2 \)?
   (a) 0 feet per second
   (b) -12 feet per second
   (c) -24 feet per second
   (d) -32 feet per second
   (e) -52 feet per second

45. While he’s in the air, when does his velocity equal 0?
   (a) when \( t = 0 \)
   (b) when \( t = \frac{1}{8} \)
   (c) when \( t = \frac{3}{8} \)
   (d) when \( t = \frac{1}{2} \)
   (e) when \( t = 2 \)
   (f) when \( t = 16 \)
   (g) when \( t = 32 \)
   (h) he’s always moving, so his velocity never equals 0

46. Let \( a(t) \) be his acceleration. Which of the following is true?
   (a) While he’s in the air, his acceleration is positive
   (b) While he’s in the air, his acceleration is negative
   (c) While he’s in the air, his acceleration is sometimes positive and sometimes negative
   (d) There is not enough information to decide.
Problems 47 through 49 are about the following situation: A spherical balloon is rising from the ground at the rate of 5 feet per second. As it rises its radius $r$ expands at the rate of 2 inches per second. When it started its radius was 1 inch. [Note: the volume of the balloon is $V = \frac{4}{3} \pi r^3$.]

47. Which of the following is true?
   (a) At 4 seconds, the balloon's height is 8 feet and its volume is between 3000 and 4000 in$^3$.
   (b) At 4 seconds, the balloon's height is 20 feet and its volume is between 5000 and 6000 in$^3$.
   (c) At 4 seconds, the balloon's height is 8 feet and its volume is between 5000 and 6000 in$^3$.
   (d) At 4 seconds, the balloon's height is 20 feet and its volume is between 3000 and 4000 in$^3$.
   (e) There is not enough information to tell.

48. Which of the following is the best approximation to the rate at which the volume of the balloon is changing 4 seconds after it started rising?
   (a) about 1014 in$^3$ per second
   (b) about 1016 in$^3$ per second
   (c) about 1018 in$^3$ per second
   (d) about 1020 in$^3$ per second
   (e) about 1022 in$^3$ per second
   (f) about 1024 in$^3$ per second

49. A rock is on the ground 15 feet away from where the balloon left the ground. How fast is the distance between the lowest point of the balloon and the rock changing 4 seconds after the balloon started rising?
   (a) 2 feet per second
   (b) 4 feet per second
   (c) 5 feet per second
   (d) 10 feet per second
   (e) 15 feet per second
   (f) 25 feet per second

50. A cubic block of ice is melting at an unnaturally controlled rate. Its volume is changing at the constant rate of -75 cm$^3$ per minute. When the length of each of its sides is 5 cm, how fast is this length changing?
   (a) 0 cm per minute
   (b) - 1 cm per minute
   (c) - 5 cm per minute
   (d) -25 cm per minute
   (e) -75 cm per minute
51. A cubic block of ice is melting in a more normal fashion. If you use differentials to approximate
the change in volume from when the length of each of its sides is 5 cm to when the length of each
of its sides is 4.9 cm, your answer is which of the following? \(^3\)
   (a) -7.35 cm\(^3\)
   (b) -7.40 cm\(^3\)
   (c) -7.45 cm\(^3\)
   (d) -7.50 cm\(^3\)
   (e) -7.55 cm\(^3\)
   (f) -7.60 cm\(^3\)
   (g) -7.65 cm\(^3\)
   (h) -7.70 cm\(^3\)

52. An ant is moving on a blackboard along a number line so that its position at time \(t\) is \(t^2 - t^3\),
where \(0 \leq t \leq 10\). What is the ant’s acceleration when \(t = 3\)?
   (a) - 21
   (b) -18
   (c) -16
   (d) -3
   (e) 3
   (f) 16
   (g) 18
   (h) 21

\(^3\)If you compute the actual change in volume you will get the wrong answer.