Basic Bridge Math

Question 1: How many bridge hands are there? That is, how many ways are there of selecting 13 cards from a deck of 52?

No problem — just start listing all the possible hands! Just kidding. That would require a ludicrously huge amount of time and scratch paper. Even computers have better things to do with their time. There must be a better way to count hands without listing every single one.

Imagine dealing a single bridge hand. For simplicity, we won’t worry about the other three players; we’ll just deal North thirteen cards.

- There are 52 possibilities for the first card. Let’s say it’s the ♦7.
- There are 51 possibilities for the second card (because it can be anything except the ♦7). Let’s say it’s the ♠K.
- There are 50 possibilities for the third card (because it can be anything except the ♦7 or ♠K). Let’s say it’s the ♣4.
- . . .
- There are 40 possibilities for the thirteenth card (because it can be anything except the ♦7, ♠K, ♣4, or any of the nine other cards already dealt).

That means that the number of possible hands is $52 \times 51 \times 50 \times \cdots \times 42 \times 41 \times 40 = 3954242643911239680000$. Right?

Close, but wrong. The reason is that the order in which the cards were dealt is irrelevant. If, say, the ♦7 and ♠K had switched places, or even if the first thirteen cards had been shuffled arbitrarily while leaving the rest of the deck alone, then the hand we dealt wouldn’t have changed.

How many ways are there to shuffle a thirteen-card hand? (To put it another way, if you accidentally drop your hand on the floor, then how many different orders can the cards be in when you pick it up again)? The first card you pick up could be any of 13; the second card could be any of 12; the third any of 11; and so on. Therefore, the answer is

$$13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6,227,020,800 = 13!.$$ 

That punctuation is not a typo. These numbers come up so often that we mathematicians use the notation “13!” (pronounced “thirteen factorial”) as a convenient shorthand rather than writing out all thirteen factors on the left.

Getting back to the original question, we need to divide the number 3954242643911239680000 by the number of ways to shuffle a 13-card hand in order to count the number of possible hands. That is,

$$\frac{52 \times 51 \times 50 \times \cdots \times 42 \times 41 \times 40}{13 \times 12 \times 11 \times \cdots \times 3 \times 2 \times 1} = \frac{3954242643911239680000}{6227020800} = 635013559600.$$ 

Also, by the way, observe that $52 \times 51 \times \cdots \times 40 = 52!/39!$ — because if you wrote $52!/39!$ out in its full glory, then you’d be able to cancel 39/39 and 38/38 and 37/37 and a lot more stuff, and all that survived
would be $52 \times 51 \times \cdots \times 40$. That means that we can write down a more concise formula:

$$63501359600 = \frac{52!}{39! \times 13!}.$$  

This kind of calculation ("How many ways are there to choose a subset of $k$ things from a set of $n$ things?") is extremely common; common enough, in fact, that it has a special symbol: \( \binom{n}{k} \). (You might also see \( C(n, k) \) or \( _nC_k \), but the notation \( \binom{n}{k} \) is what most mathematicians would use.) By the same logic as above, these numbers can be calculated by the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)} = \frac{n!}{k!(n-k)!}$$

So we can summarize: The number of possible bridge hands is

$$\binom{52}{13} = 63501359600.$$