Problem: Find all continuous functions $f$ on $(0, \infty)$ such that

$$\int_x^{x^2} f(t) \, dt = \int_1^x f(t) \, dt$$

for all $x > 0$.

Solution: Differentiating both sides gets you $2xf(x^2) - f(x) = f(x)$, which simplifies to $xf(x^2) = f(x)$. First, by inspection, $f(x) = c/x$ satisfies this equation for any constant $c$. How do we prove that there are no other solutions? By the following trick:

Rewrite the equation $xf(x^2) = f(x)$ as $f(x^{1/2}) = f(x)/x$. This is true for all positive $x$, so we could replace $x$ with any of $x^{1/2}$, $x^{1/4}$, ... , to obtain the equations

(1) $f(x^2) = f(x)/x$,
(2) $f(x) = f(x^{1/2})/x^{1/2}$,
(3) $f(x^{1/2}) = f(x^{1/4})/x^{1/4}$,
(4) $f(x^{1/4}) = f(x^{1/8})/x^{1/8}$,

... Plug (3) into (2) to get

$$f(x) = \frac{f(x^{1/2})}{x^{1/2}} = \frac{f(x^{1/4})}{x^{3/4}}.$$

Plug (4) into this equation to get

$$f(x) = \frac{f(x^{1/8})}{x^{3/8}} = \frac{f(x^{1/8})}{x^{7/8}}.$$

Repeating this procedure will produce a sequence of equations of the form

(5) $f(x^{1/2^n}) = x^{(1-1/2^n)}f(x)$

for all positive integers $n$. Everything in sight is continuous, so we can take the limit as $n \to \infty$ of both sides of (5), giving

$$f(1) = xf(x)$$

for all $x$, which says that $f(x) = f(1)/x$. In other words, if $f$ is a function that is a solution, then let $c = f(1)$; then $f(x) = c/x$. So every solution is of the form $f(x) = c/x$. 