Problems for the First KS math competition

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• Problem 1
Let \( f \) be a continuous function on \([0, 1]\), such that for every \( x \in [0, 1] \), \( \int_{x}^{1} f(t) dt \geq \frac{1-x^2}{2} \). Show that

\[
\int_{0}^{1} f^2(x) dx \geq \frac{1}{3}.
\]

Solution:

\[
0 \leq \int_{0}^{1} (f(x) - x)^2 dx = \int_{0}^{1} f^2(x) dx - 2 \int_{0}^{1} xf(x) dx + \int_{0}^{1} x^2 dx.
\]

It follows

\[
\int_{0}^{1} f^2(x) dx \geq 2 \int_{0}^{1} xf(x) dx - \frac{1}{3}.
\]

But

\[
\frac{1}{3} = \int_{0}^{1} \frac{1-x^2}{2} dx \leq \int_{0}^{1} (\int_{0}^{x} f(t) dt) dx = \int_{0}^{1} x f(t) dt,
\]

whence

\[
\int_{0}^{1} f^2(x) dx \geq \frac{1}{3}.
\]
**Problem 2** Let \( x_{n+1} = \frac{4}{2 - x_n} \), where \( x_0 = 1 \). Determine \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k \).

**Solution** The sequence is periodic with period 3: \( x_0 = 1, x_1 = 4, x_2 = -2 \) and \( x_3 = 1 \). It follows that \( S_n = \sum_{k=1}^{n} x_k \) is

\[
S_n = \begin{cases} 
3m & n = 3m \\
3m + 4 & n = 3m + 1 \\
3m + 2 & n = 3m + 2
\end{cases}
\]

It is clear that \( 1 \leq S_n/n \leq (n + 3)/n \) and the \( \lim_{n \to \infty} S_n/n = 1 \).

**Problem 3** Let \( P \) be a polynomial of degree \( n \) with real coefficients and real zeros only. Show that

\[
(n - 1)(P'(x))^2 \geq nP(x)P''(x).
\]

When do you achieve equality for all \( x \)? **Solution:** Since \( P(x) = a(x-x_1) \ldots (x-x_n) \), we have

\[
\frac{P'(x)}{P(x)} = \sum_{j=1}^{n} \frac{1}{x-x_j} \\
\frac{P''(x)}{P(x)} = \sum_{1 \leq i < j \leq n} \frac{2}{(x-x_j)(x-x_i)}
\]

Thus

\[
(n - 1) \left( \frac{P'(x)}{P(x)} \right)^2 - n \frac{P''(x)}{P(x)} = \sum_{j=1}^{n} \frac{(n - 1)}{(x-x_j)^2} - \sum_{1 \leq i < j \leq n} \frac{2}{(x-x_j)(x-x_i)} = \\
= \sum_{1 \leq i < j \leq n} \left( \frac{1}{x-x_i} - \frac{1}{x-x_j} \right)^2 \geq 0.
\]
• Problem 4
Find all differentiable functions \( F : \mathbb{R}^+ \to \mathbb{R}^+ \), so that
\[
f(x)f(yf(x)) = f(x + y).
\]

Solution:
Write the condition as
\[
f^2(x) \frac{f(yf(x)) - 1}{yf(x)} = \frac{f(x + y) - f(x)}{y}
\]
Take a limit as \( y \to 0 \) to get \( f'(x) = -f'(0)f^2(x) \), which gives the solution \( f(x) = \frac{1}{(ax + b)} \). Plug this in the original equation to find that only when \( b = 1 \), this will be satisfied.

• Problem 5
Let \( A \) and \( B \) are two \( n \times n \) symmetric matrices with real entries, which do not necessarily commute. Assume also that \( A \) is positive in the sense that all eigenvalues are positive. Show that \( AB \) has all eigenvalues real.

Solution: Since \( A \) is symmetric and positive, then \( A = T^{-1}KT \), where \( K \) is diagonal with positive entries \( \lambda_1, \ldots, \lambda_n \) on the diagonal and \( T \) is invertible matrix. Define \( K_{1/2} \), to be the diagonal matrix with \( \sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n} \) on the diagonal and \( C = T^{-1}K_{1/2}T \) is invertible. Clearly \( K_{1/2}^2 = K \) and \( C^2 = A \). We have
\[
C^{-1}ABC = C^{-1}C^2BC = CBC.
\]
It is clear that \( CBC \) is symmetric with real entries \( (CBC)^t = C^tB^tC^t = CBC \) and therefore has only real eigenvalues. But \( AB \) is similar to \( CBC \) and therefore has the same real eigenvalues.
Problem 6
Let $A$ be a real $4 \times 2$ matrix, while $B$ is real $2 \times 4$ matrix. We know
\[
AB = \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}
\]
Find $BA$.
Solution:
Represent $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $B = (B_1, B_2)$, where $A_1, A_2, B_1, B_2$ are $2 \times 2$ matrices. We have
\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} (B_1, B_2) = \begin{pmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{pmatrix}.
\]
It follows that $A_1 B_1 = A_2 B_2 = I$ and $A_1 B_2 = A_2 B_1 = -I$. Then
\[
BA = (B_1, B_2) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = B_1 A_1 + B_2 A_2 = 2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.
\]

Problem 7
Let $p_1, \ldots, p_n$ be finitely many points in the unit ball. Show that there exists at least one point on the unit circle $p$, so that
\[
\frac{1}{n} \sum_{k=1}^{n} |p - p_i| \geq 1.
\]
Solution: Choose $p$ to be the unit vector in the direction opposite to $p_1 + \ldots + p_n$. We have by the triangle inequality
\[
\sum_{j=1}^{n} |p - p_j| \geq |np - \sum_{j=1}^{n} p_j| = n + \sum_{j=1}^{n} p_j | \geq n.
\]
• Problem 8

Let $z \neq 0$ and $A$ and $B$ are two matrices, with

$$AB - BA = zA$$

Show that for all integers $k$, $A^kB - BA^k = zkA^k$. Show that there exists $k$, so that $A^k = 0$.

Solution: We have

$$A^kB - BA^k = \sum_{j=1}^{k} (A^{k-j+1}BA^{j-1} - A^{k-j}BA^j) =$$

$$= \sum_{j=1}^{k} A^{k-j}(AB - BA)A^{j-1} = \sum_{j=1}^{k} A^{k-j}zAA^{j-1} = zkA^k.$$

For the second part, it is equivalent to show that $A$ has only zero eigenvalues. Suppose not. Assume without loss of generality (by rescaling) that $A$ has eigenvalues, satisfying $|\lambda| \leq 1$ and an eigenvalue $\lambda_0 : |\lambda_0| = 1$. It is clear now that the entries of $A^k$ are uniformly bounded in $k$, whence the entries of $A^kB - BA^k$ are uniformly bounded in $k$. The right-hand $zkA^k$ has entries that increase linearly with $k$ and that is a contradiction.