Third KS math competition

April 3, 2007

1. Show that for every sequence \( x_1, \ldots, x_n \in (0, 1) \) at least one of the inequalities holds:
   \[
x_1 \ldots x_n \leq 2^{-n}
   \]
or
   \[
   (1 - x_1) \ldots (1 - x_n) \leq 2^{-n}
   \]

   **Solution:** Suppose the claim is not true. Then
   \[
x_1 \ldots x_n (1 - x_1) \ldots (1 - x_n) > 4^{-n}
   \]
   But \( x_1(1 - x_1) \leq 1/4, \ldots, x_n(1 - x_n) \leq 1/4 \), we get
   \[
   4^{-n} < x_1 \ldots x_n(1 - x_1) \ldots (1 - x_n) \leq 4^{-n},
   \]
a contradiction.

2. Compute
   \[
   L = \lim_{n \to \infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n}
   \]

   **Solution:** Let \( A_n = n^{-4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n} \). We have
   \[
   \ln(A_n) = \left( \sum_{i=1}^{2n} \frac{1}{n} \ln(n^2 + i^2) \right) - 4 \ln(n) = \left( \sum_{i=1}^{2n} \frac{1}{n} \ln(n^2(1 + i^2/n^2)) \right) - 4 \ln(n) = \left( \sum_{i=1}^{2n} \frac{1}{n} (2 \ln(n) + \ln(1 + i^2/n^2)) \right) - 4 \ln(n) = \sum_{i=1}^{2n} \frac{1}{n} \ln(1 + i^2/n^2)
   \]
The last expression is a Riemann sum for \( \int_0^2 \ln(1+x^2)dx \) and therefore \( \lim_n A_n = e^{\int_0^2 \ln(1+x^2)dx} \) and

\[
\int_0^2 \ln(1+x^2)dx = x \ln(1+x^2)\big|_0^2 - 2 \int_0^2 \frac{x^2}{1+x^2}dx = 2 \ln(5) - 2 + 2 \tan^{-1}(2).
\]

Thus \( L = e^{2 \ln(5)-2+2 \tan^{-1}(2)} = 25 e^{2 \tan^{-1}(2)-2} \).

3. Find the limit of the sequence

\[
\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \ldots
\]

**Solution:** Denote the \( n \)th term of the sequence by \( x_n \). We have \( x_{n+1} = \sqrt{1+x_n} \), \( x_1 = 1 \). We establish by induction that \( x_{n+1} > x_n \). Indeed, \( x_2 = \sqrt{2} > x_1 = 1 \).

Assuming \( x_n > x_{n-1} \), we get

\[
x_{n+1} = \sqrt{1+x_n} > \sqrt{1+x_{n-1}} = x_n.
\]

Next, we establish by induction that \( x_n < 2 \) (Any number bigger than \( (1+\sqrt{5})/2 \) will do here). Indeed, \( x_1 < 2 \) and assuming \( x_n < 2 \), we get \( x_{n+1} = \sqrt{1+x_n} < \sqrt{3} < 2 \).

Thus, \( x_n \) is increasing and bounded, thus convergent. Denote \( L = \lim_n x_n \). Then \( L = \sqrt{1+L} \), whence \( L = (1+\sqrt{5})/2 \).

4. A coin is tossed 10 times. Find the probability of not having two consecutive tails.

**Solution:** Solve more general problem with \( n \) tosses instead of 10 tosses. Record the outcomes with 0 if tail turns up and 1 otherwise. Thus, we are counting the numbers of sequences of 0, 1, which will not have two consecutive 0. Denote the number of these (let us call them favorable sequences) \( f_n \). Any sequence like that will either start with zero or one. If it starts with 1, we may concatenate this with any favorable sequence of length \( n-1 \). If it starts with a zero, then we must have 1 in the next slot, after which, we may concatenate with any favorable sequence with length \( n-2 \). Thus \( f_n = f_{n-1} + f_{n-2} \). Also, see that \( f_1 = 2, f_2 = 3 \). Thus, \( f_{10} = 144 \). The total number of outcomes is \( 2^{10} = 1024 \), and thus the probability is 144/1024.

5. How many squares (of all possible sizes) are there in the following picture?
Solution: Count the number of possible lower left end corners of squares with sidelength $k, 1 \leq k \leq 10$. We have clearly the lower left square with sidelength $(10 - k)$, this yields $(10 - k + 1)^2$ points. Thus the number of squares is

$$\sum_{k=1}^{10} (10 - k + 1)^2 = 385.$$