THE 28th ANNUAL MATHEMATICS PRIZE COMPETITION
(SENIOR)
Department of Mathematics
University of Kansas
April 20, 2010

Instructions

1. **DO NOT** write your name on your solutions; use the last three digits of your KUID. Write this number on each solution that you submit.

2. Submit each solution on a separate sheet. If a solution takes more than one page, number the pages and staple them together. (*Solutions of different problems should not be stapled together.*)

3. You need not hand in your scratch work.

4. You may keep this sheet of problems.

5. The only aid permitted is a translation dictionary — no calculators or other notes.

6. Each of the 6 problems is worth 10 points. If a problem has more than one part, the parts do not necessarily have equal weight. Partial credit may be given for significant progress towards a solution.

7. Winners will be notified by mail. The prizes will be awarded at the Mathematics Department Banquet on Tuesday, April 27.

8. Have fun!
S1. Let $P, Q, R, S$ be polynomials. Prove that

$$\left( \int_1^x P(t)R(t)dt \right) \left( \int_1^x Q(t)S(t)dt \right) - \left( \int_1^x P(t)S(t)dt \right) \left( \int_1^x Q(t)R(t)dt \right)$$

is divisible by $(x - 1)^4$.

S2. How many different ways are there to walk from square A to square B in the following diagram, using the fewest possible number of steps? You can only walk to a square sharing a side (not just a corner) with the square you are currently on. The dotted line shows one possible route.

![Diagram of a grid with square A at the bottom left and square B at the bottom right.](image)

S3. Find a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'''(x) = f(x)$ but $f'(x) \neq f(x)$.

S4. Prove that the sum of two or more consecutive perfect cubes (e.g., $27 + 64 + 125$) cannot be prime.

S5. Switches $s_1, \ldots, s_n$ control lightbulbs $b_1, \ldots, b_n$ by toggling them off and on. For every $i$, switch $s_i$ controls lightbulb $b_i$ and possibly others, but $s_i$ controls $b_j$ if and only if $s_j$ controls $b_i$. Right now, the lights are all off. Prove that it is possible to turn them all on.

S6. Two players play the following game. The game board is a $3 \times 3$ matrix $M$ which is initially empty. Player #1 starts by writing a 1 in the matrix. Then Player #2 writes a 2 in the matrix. The players keep taking turns in this way, with Player #1 writing a 1 every time and player #2 writing a 2 every time, until the matrix is completely filled (so the matrix ends up with five 1’s and four 2’s. If $\det(M) \neq 0$ at the end of the game, then Player #1 wins; otherwise, Player #2 wins.

You’re going to play the game with Prof. Martin. Come up to the front of the room when you’re ready to play. You get to pick who is Player #1 and who is Player #2. You get two tries. Winning the game on your first try is worth 10 points, and on your second try try 5 points.