Complete Graphs

Let \( N \) be a positive integer.

**Definition:** A complete graph is a graph with \( N \) vertices and an edge between every two vertices.

- There are no loops.
- Every two vertices share exactly one edge.

We use the symbol \( K_N \) for a complete graph with \( N \) vertices.
Complete Graphs

$K_1$

$K_2$

$K_3$

$K_4$

$K_5$

$K_6$
How many edges does $K_N$ have?
Complete Graphs

How many edges does $K_N$ have?

- $K_N$ has $N$ vertices.
How many edges does $K_N$ have?

- $K_N$ has $N$ vertices.
- Each vertex has degree $N - 1$. 

The number of edges in $K_N$ is $N(N - 1)/2$. 

Now, the Handshaking Theorem tells us that...
Complete Graphs

How many edges does $K_N$ have?

- $K_N$ has $N$ vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$. 

Now, the Handshaking Theorem tells us that the number of edges in $K_N$ is $\frac{N(N - 1)}{2}$. 
How many edges does $K_N$ have?

- $K_N$ has $N$ vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$.
- Now, the Handshaking Theorem tells us that...
Complete Graphs

How many edges does $K_N$ have?

- $K_N$ has $N$ vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$.
- Now, the Handshaking Theorem tells us that...

The number of edges in $K_N$ is $\frac{N(N - 1)}{2}$. 


The number of edges in $K_n$ is $\frac{n(n-1)}{2}$.

This formula also counts the number of pairwise comparisons between $N$ candidates (recall §1.5).
The number of edges in $K_N$ is $\frac{N(N - 1)}{2}$.

- This formula also counts the number of pairwise comparisons between $N$ candidates (recall §1.5).
- The Method of Pairwise Comparisons can be modeled by a complete graph.
The number of edges in $K_N$ is $\frac{N(N - 1)}{2}$.

- This formula also counts the number of pairwise comparisons between $N$ candidates (recall §1.5).
- The Method of Pairwise Comparisons can be modeled by a complete graph.
  - Vertices represent candidates
  - Edges represent pairwise comparisons.
  - Each candidate is compared to each other candidate.
  - No candidate is compared to him/herself.
How many different Hamilton circuits does $K_N$ have?

- Let’s assume $N = 3$. 

How many different Hamilton circuits does $K_N$ have? 

- Let’s assume $N = 3$.

- We can represent a Hamilton circuit by listing all vertices of the graph in order.

- The first and last vertices in the list must be the same. All other vertices appear exactly once.

- We’ll call a list like this an “itinerary”.
Hamilton Circuits in $K_N$

How many different Hamilton circuits does $K_N$ have?

Some possible itineraries:

- $A, C, D, B, A$
- $Y, X, W, U, V, Z, Y$
- $Q, W, E, R, T, Y, Q$

- The first/last vertex is called the “reference vertex”.

Hamilton Circuits in $K_N$

How many different Hamilton circuits does $K_N$ have?

Some possible itineraries:

- $A, C, D, B, A$
- $Y, X, W, U, V, Z, Y$
- $Q, W, E, R, T, Y, Q$

- The first/last vertex is called the “reference vertex”.

- **Changing the reference vertex does not change the Hamilton circuit**, because the same edges are traveled in the same directions.

- That is, different itineraries can correspond to the same Hamilton circuit.
Changing the reference vertex does not change the Hamilton circuit.

For example, these itineraries all represent the same Hamilton circuit in $K_4$:

- $A, C, D, B, A$ (reference vertex: $A$)
- $B, A, C, D, B$ (reference vertex: $B$)
- $D, B, A, C, D$ (reference vertex: $C$)
- $C, D, B, A, C$ (reference vertex: $D$)
Changing the reference vertex does not change the Hamilton circuit.

For example, these itineraries all represent the same Hamilton circuit in $K_4$:

- $A, C, D, B, A$ (reference vertex: $A$)
- $B, A, C, D, B$ (reference vertex: $B$)
- $D, B, A, C, D$ (reference vertex: $C$)
- $C, D, B, A, C$ (reference vertex: $D$)

Every Hamilton circuit in $K_N$ can be described by exactly $N$ different itineraries (since there are $N$ possible reference vertices).
So, how many possible itineraries are there?
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- ...
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- . . .
- 2 possibilities for the $(N - 1)st$ vertex
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- ... 
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the $N$th vertex
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- . . .
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the $N$th vertex
- and then the reference vertex again.
So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- \ldots
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the $N$th vertex
- and then the reference vertex again.

If we are counting Hamilton circuits, then we don’t care about the reference vertex.
Conclusion: The number of Hamilton circuits in $K_N$ is

$$(N - 1) \times (N - 2) \times \cdots \times 3 \times 2 \times 1 = (N - 1)!$$

Each one can be described by $N$ different itineraries.

(So the number of itineraries is actually $N!$.)
For every $N \geq 3$,

\[
\text{The number of Hamilton circuits in } K_N \text{ is } (N - 1)!.
\]

In comparison, for every $N \geq 1$,

\[
\text{The number of edges in } K_N \text{ is } \frac{N(N - 1)}{2}.
\]
## Hamilton Circuits in $K_N$

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Edges $N(N-1)/2$</th>
<th>Hamilton circuits $(N-1)!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>120</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
<td>620</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>16</td>
<td>120</td>
<td>1307674368000</td>
</tr>
</tbody>
</table>
Hamilton Circuits in $K_3$

Itineraries in $K_3$:

<table>
<thead>
<tr>
<th>A, B, C, A</th>
<th>A, C, B, A</th>
</tr>
</thead>
<tbody>
<tr>
<td>B, C, A, B</td>
<td>B, A, C, B</td>
</tr>
<tr>
<td>C, A, B, C</td>
<td>C, B, A, C</td>
</tr>
</tbody>
</table>
Hamilton Circuits in $K_3$

Itineraries in $K_3$: 

| A, B, C, A | A, C, B, A |
| B, C, A, B | B, A, C, B |
| C, A, B, C | C, B, A, C |

- Each column of the table gives 3 itineraries for the same Hamilton circuit (with different reference vertices).
- The number of Hamilton circuits is $(3 - 1)! = 2! = 2$. 
# Hamilton Circuits in $K_4$

## Itineraries in $K_4$:

<table>
<thead>
<tr>
<th>ABCDA</th>
<th>ABDCA</th>
<th>ACBDA</th>
<th>ACDBA</th>
<th>ADBCA</th>
<th>ADCBA</th>
</tr>
</thead>
<tbody>
<tr>
<td>BCDAB</td>
<td>BDCAB</td>
<td>BDACB</td>
<td>BACDB</td>
<td>BCADB</td>
<td>BADCB</td>
</tr>
<tr>
<td>CDABC</td>
<td>CABDC</td>
<td>CBDAC</td>
<td>CDBAC</td>
<td>CADBC</td>
<td>CBADC</td>
</tr>
<tr>
<td>DABCD</td>
<td>DCABD</td>
<td>DACBD</td>
<td>DBACD</td>
<td>DBCAD</td>
<td>DCBAD</td>
</tr>
</tbody>
</table>

- Each column lists 4 itineraries for the same Hamilton circuit.
- The number of Hamilton circuits is $(4 - 1)! = 3! = 6$. 
Where have you seen this table before?
Hamilton Circuits in $K_4$

Where have you seen this table before?

<table>
<thead>
<tr>
<th>ABCD</th>
<th>ABDC</th>
<th>ACBD</th>
<th>ACDB</th>
<th>ADBC</th>
<th>ADCB</th>
</tr>
</thead>
<tbody>
<tr>
<td>BCDA</td>
<td>BDCA</td>
<td>BDAC</td>
<td>BACD</td>
<td>BCAD</td>
<td>BADC</td>
</tr>
<tr>
<td>CDAB</td>
<td>CABD</td>
<td>CBDA</td>
<td>CDBA</td>
<td>CADB</td>
<td>CBAD</td>
</tr>
<tr>
<td>DABC</td>
<td>DCAB</td>
<td>DACB</td>
<td>DBAC</td>
<td>DBCA</td>
<td>DCBA</td>
</tr>
</tbody>
</table>
Hamilton Circuits in $K_4$

Where have you seen this table before?

<table>
<thead>
<tr>
<th>ABCD</th>
<th>ABDC</th>
<th>ACBD</th>
<th>ACDB</th>
<th>ADBC</th>
<th>ADCB</th>
</tr>
</thead>
<tbody>
<tr>
<td>BCDA</td>
<td>BDCA</td>
<td>BDAC</td>
<td>BACD</td>
<td>BCAD</td>
<td>BADC</td>
</tr>
<tr>
<td>CDAB</td>
<td>CABD</td>
<td>CBDA</td>
<td>CDBA</td>
<td>CADB</td>
<td>CBAD</td>
</tr>
<tr>
<td>DABC</td>
<td>DCAB</td>
<td>DACB</td>
<td>DBAC</td>
<td>DBCA</td>
<td>DCBA</td>
</tr>
</tbody>
</table>

An itinerary (without the last vertex repeated) is the same thing as the list of sequential coalitions in a weighted voting system!

That’s why there are $N!$ itineraries on $N$ vertices.
By the way, for which values of $N$ does the complete graph $K_N$ have an Euler circuit?
By the way, for which values of $N$ does the complete graph $K_N$ have an Euler circuit?

Answer: When $N$ is odd. (Every vertex in $K_N$ has degree $N - 1$, so we need $N - 1$ to be even.)