#1 Let $G = (V, E)$ be a connected graph, and let $T \subseteq E$. Prove that $T$ is a spanning tree of $G$ if and only if at least two of the following conditions are met:

- $T$ spans every vertex of $G$.
- $T$ is acyclic.
- $|T| = |V| - 1$.

#2 Complete the sentence:

“An edge $e \in E$ belongs to every spanning tree of $G = (V, E)$ if and only if $e$ is ____________.”

#3 Prove the deletion-contraction formula

$$
\tau(G) = \tau(G - e) + \tau(G/e)
$$

for any graph $G$ and any edge $e$, by verifying that there are (natural) bijections

$$
\{T \in T(G) \mid e \notin T\} \leftrightarrow T(G - e)
$$

and

$$
\{T \in T(G) \mid e \in T\} \leftrightarrow T(G/e).
$$

How does this formula tell you something about Problem #2?

#4 Prove the Matrix-Tree Theorem using (1) and facts about determinants.

#5 Count the number of spanning trees of the graph

in three ways: (i) with your bare hands; (ii) with the deletion-contraction formula (1); (iii) using the Matrix-Tree Theorem.

#6 Suppose that $G$ is not connected. What happens if you try to calculate $\tau(G)$ using the Matrix-Tree Theorem?

#7 Prove that Prüfer coding is a bijection. (As mentioned in the notes, the essence of this problem is figuring out a way to run the algorithm in reverse—that is, for a code $(v_1, \ldots, v_n) \in [n]^{n-2}$, compute the spanning tree $P^{-1}(T) \in T(K_n)$.)

#8 Let $G$ be a graph and $q$ a positive integer. Define $G^{(q)}$ to be the graph in which each edge of $G$ is replaced with $q$ copies of itself. Derive a formula for $\tau(G^{(q)})$ in terms of $\tau(G)$. 
#9  (Hard!) Use problem #8 to derive the weighted formula

$$\sum_{T \in T(Q_n)} \text{wt}(T) = 2^{2^n-n-1}q_1 \cdots q_n \prod_{S \subseteq [n], |S| \geq 2} \left( \sum_{t \in S} q_t \right)$$

from the unweighted (weaker) formula

(2)  \( \tau(Q_n) = \prod_{S \subseteq [n], |S| \geq 2} 2|S| \).

#10  Prove (2) bijectively, publish a paper with your result, and be sure to mention Mathcamp '04 in the "Thanks" section.