1. The Chromatic Polynomial

A \textit{k-coloring} of a graph \(G = (V, E)\) is nothing more nor less than a function \(f : V(G) \to [k]\). It’s a \textit{proper k-coloring} if no two adjacent vertices get the same color, that is,

\[ f(v) \neq f(w) \quad \text{for} \quad v \sim w. \]

Of course, for a given graph \(G\) and integer \(k\), there is no guarantee that \(G\) has any proper \(k\)-colorings at all! So it makes sense to study the \textit{chromatic function} defined by

\[ P_G(k) = \text{number of proper } k\text{-colorings of } G. \]

Sometimes this is pretty easy to determine. For a really stupid example, suppose that \(G\) has no edges. Then no two vertices are adjacent, and every coloring is proper, so the chromatic function is \(P_G(k) = k^n\), where \(n = |G|\). More generally, if \(G\) is disconnected, then a proper \(k\)-coloring of \(G\) is the same thing as a proper \(k\)-coloring of each of its components, so \(P_G(k)\) is the product of their chromatic functions. Another stupid example is the case that \(G\) has one or more loops. Then \(P_G(k) = 0\), because the two endpoints of a loop will certainly always get the same color.

Much less trivially, suppose that \(G = K_n\). To construct a proper \(k\)-coloring \(f\) of \(K_n\), we can start by coloring vertex 1 arbitrarily, which gives a factor of \(k\) in \(P_G(k)\). We then have \(k - 1\) choices for the color of vertex 2 (since we cannot use \(f(1)\)), \((k - 2)\) choices for the color of vertex 3 (since we cannot use \(f(1)\) or \(f(2)\)), \(\ldots\), \((k - n + 1)\) choices for the color of vertex \(n\). It follows that

\[ P_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1). \]

Note that this is zero if and only if \(k < n\), which certainly makes sense.

**Theorem:** Let \(G = (V, E)\) be a simple graph. Then the chromatic function \(P_G(k)\) is actually a polynomial in \(k\), defined recursively by the rules

\begin{enumerate}
\item \(P_G(k) = 0\) if \(G\) has one or more loops,
\item \(P_G(k) = k\) if \(G\) consists of a single vertex,
\item \(P_G(k) = P_{G_1}(k) \cdots P_{G_s}(k)\) if \(G_1, \ldots, G_s\) are the connected components of \(G\),
\item \(P_G(k) = P_{G-e}(k) - P_{G/e}(k)\) for any nonloop \(e \in E(G)\).
\end{enumerate}

We’ve already discussed the three base cases; the actual recurrence is left as an exercise (it’s not hard). Note that the form of the recurrence implies by induction that \(P_G(k)\) is a polynomial in \(k\), so we drop the term “chromatic function” and use “chromatic polynomial” from here on in.

The recurrence (4) is eerily reminiscent of the deletion-contraction recurrence for spanning trees, namely

\[ \tau(G) = \tau(G - e) + \tau(G/e). \]

2. Acyclic and Totally Cyclic Orientations

An \textit{orientation} of \(G = (V, E)\) is a choice of a “head” and a “tail” for each edge of \(G\). We can represent such a thing by drawing an arrow from the tail to the head. (So an orientation is really a directed graph (“digraph”) \(D\) whose underlying undirected graph is \(G\).)

A \textit{directed cycle} in a digraph is a sequence of vertices \((v_1, \ldots, v_s)\) such that for every \(i \in [s]\), there is an edge with tail \(v_i\) and head \(v_{i+1}\). (The indices are taken modulo \(n\), so that \(v_{s+1} = v_1\).)
An orientation is **acyclic** if it contains no directed cycles, and is **totally cyclic** if every edge belongs to at least one directed cycle.

\[
\begin{align*}
\text{Undirected graph} & \quad \text{Acyclic orientation} & \quad \text{Totally cyclic orientation} & \quad \text{Neither acyclic nor totally cyclic} \\
\end{align*}
\]

Define

\[
\begin{align*}
ao(G) & = \text{number of acyclic orientations of } G, \\
tc(G) & = \text{number of totally cyclic orientations of } G.
\end{align*}
\]

(Why might anyone care about these things? Well, partially ordered sets are something that combinatorialists care a great deal about, and an acyclic orientation of \( G \) is the very same thing as a partial order on \( V(G) \) in which every pair of adjacent vertices is comparable. How about totally cyclic orientations? Well, consider a random walk on a directed graph in which each step consists of randomly choosing an edge whose tail is your current location, and walking to the head of that edge. If the graph is totally cyclic, then the walk will go on forever; otherwise, you will almost always get stuck.)

It is pretty easy to show that \( \ao(G) > 0 \) if and only if \( G \) has no loops, and that \( tc(G) > 0 \) if and only if \( G \) has no bridge. (A *bridge*, also known as a *cut-edge*, *isthmus*, or *coloop*, is an edge \( e \) such that \( G - e \) has more components than \( G \).

However, how do we calculate \( \ao(G) \) and \( tc(G) \) in general? Once again, there are deletion-contraction recurrences:

\[
\begin{align*}
\ao(G) & = \ao(G - e) + \ao(G/e), \\
tc(G) & = tc(G - e) + tc(G/e).
\end{align*}
\]

These recurrences are slightly more subtle than those for \( \tau(G) \) and \( P_G(k) \), but they're really not hard. The trick to proving (6) is as follows. Let there be given an acyclic orientation \( D \) of \( G \) and an arbitrary edge \( e \in E(G) \). Construct a new orientation \( D' \) from \( D \) by reversing the head and tail of \( e \), while leaving all other edge directions unchanged. Classify \( D \) as “blue” or “red” depending on whether or not \( D' \) is acyclic, and then relate the numbers of blue and red acyclic orientations of \( G \) to \( \ao(G - e) \) and \( \ao(G/e) \). Something similar ought to work for \( tc(G) \), although to be honest I haven’t thought too hard about it.

### 3. The Tutte Polynomial

What is going on here? There is no obvious connection between the invariants \( \tau(G) \), \( P_G(k) \), \( \ao(G) \) and \( tc(G) \)—yet each satisfies a very similar recurrence involving deletion and contraction.

What they have in common as that they are all special cases of the **Tutte polynomial** of \( G \). This is a wonderful invariant of \( G \) which is defined precisely so as to generalize ALL invariants defined by deletion-contraction recurrences. Not only does it tie together spanning trees, chromatic polynomial, acyclic and totally cyclic orientations, but there are a host of other important graph invariants that can be obtained from the Tutte polynomial. There are even connections to knot theory and the Jones polynomial, about which I know sadly little...
**Definition:** Let $G = (V, E)$ be a graph. The **Tutte polynomial** of $G$ is the bivariate polynomial $T_G(x, y)$ given by the recurrence

$$T_G(x, y) = \begin{cases} 
1 & \text{if } E = \emptyset, \\
T_{G'}(x, y) \cdot T_{G''}(x, y) & \text{if } G \text{ is the disjoint union of } G' \text{ and } G'', \\
x \cdot T_{G/e}(x, y) & \text{if } e \text{ is a bridge}, \\
y \cdot T_{G-e}(x, y) & \text{if } e \text{ is a loop}, \\
T_{G-e}(x, y) + T_{G/e}(x, y) & \text{if } e \text{ is neither a bridge nor a loop}.
\end{cases}$$

(8)

It is not obvious from the definition that the Tutte polynomial is well-defined, since we need to check that it is independent of the choice of $e$ in the last case of (8). The best way to do this is to give another interpretation of the Tutte polynomial which is well-defined for obvious reasons, and then to prove that the two descriptions are equivalent.

For an edge set $F \subset E$, define the **rank** of $F$, denoted $r(F)$, to be the size of a spanning tree of $F$.

**Theorem:** Let $G = (V, E)$ be a graph. The Tutte polynomial of $G$ is given by the formula

$$T_G(x, y) = \sum_{F \subset E} (x-1)^{r(E)-r(F)} (y-1)^{|F|-r(F)}.$$  

(9)

It is clear that the right-hand side of (9) is well-defined; the proof is, as usual, left as an exercise. (Hint: Induct on $|E(G)|$.) By the way, the quantity $r(E) - r(F)$ is called the **corank** of $F$, and $|F| - r(F)$ is its **nullity**.