Mathematics of Voting — Lecture Notes II (Thursday)

**Fairness criteria:** Two old ones...

- **Condorcet Criterion (CC):** If there is a Condorcet winner—that is, a candidate who would defeat any other candidate in a head-to-head election—then the voting system ought to declare that candidate the winner.

(Note: The “Majority Criterion”—that a candidate who receives a majority of the first-place votes ought to win the election—is actually strictly weaker than this, because a majority candidate is automatically a Condorcet winner. However, there are voting systems which satisfy the Majority Criterion, but not CC.)

- **Independence of Irrelevant Alternatives (IIA):** Whether A beats B or B beats A should not depend on whether C is in the race. Precisely, if A is ranked above B, and C is disqualified, then A should still be ranked above B.

(A famous example of a violation of IIA: Olympic figure skating in 2002. Michelle Kwan was in first place and Sarah Hughes in second place, with both skaters having finished their programs. However, Irina Slutskaya’s performance interchanged the relative order of Kwan and Hughes, so that Hughes won the gold, Slutskaya the silver, and Kwan the bronze medal. Had Slutskaya done much worse, Kwan would have finished ahead of Hughes. (See the link on the course webpage.)

...and some new ones:

- **Monotonicity Criterion (MC):** Let A be a candidate. If one or more ballots are modified solely to favor A—that is, A moves up in the ranking on these ballots, while the relative order of other candidates is unchanged — then the final ranking of A should not go down.

For example, changing a ballot marked C,B,F,A,E,D to C,A,B,F,E,D is okay, but not to C,A,F,B,E,D, because on this last ballot the relative order of B and F has been changed.

- **Unanimity Criterion (UC) (a.k.a. weak Pareto criterion):** If every voter prefers A to B, then A should be ranked above B. (Duh.)

The simplest voting system of all is called a dictatorship: when there is only one voter whose preferences matter! This probably is not an optimal system from a democratic point of view, but it does (trivially) satisfy all of the fairness criteria we have listed.

Let’s make some precise notation and definitions:

<table>
<thead>
<tr>
<th>English</th>
<th>Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of candidates</td>
<td>$N$</td>
</tr>
<tr>
<td>Set of all candidates</td>
<td>$C = {c_1, \ldots, c_N}$</td>
</tr>
<tr>
<td>Ballot</td>
<td>Permutation (i.e., total ordering) $\sigma$ of $K$ where $\sigma(i) = i$th favorite choice</td>
</tr>
<tr>
<td>Set of all ballots</td>
<td>$S_K$ (cardinality $N!$)</td>
</tr>
<tr>
<td>Preference schedule (or profile)</td>
<td>a function $P : S_K \rightarrow \mathbb{N}$ where $P(\sigma) =$ number of voters who cast ballot $\sigma$</td>
</tr>
<tr>
<td>Voting system</td>
<td>a function $V : {\text{preference schedules}} \rightarrow S_K$</td>
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So, for example, the number of first-place votes received by candidate \( c_k \) is

\[ FPV(k) = \sum_{\sigma \in S_k \atop \sigma(1) = c_k} P(\sigma) \]

and if we define \( V(k) = j \) if \( c_k \) has the \( j \)th highest value of \( FPV \), then \( V \) is plurality voting. (So the winner is the candidate \( c_k \) such that \( V(c_k) = 1 \).)

**Arrow’s Theorem:** If \( N \geq 3 \), then any voting system \( V \) which satisfies both IIA and UC is a dictatorship!

This is fairly disturbing, as is the following related result:

**Gibbard-Satterthwaite Theorem:** If \( N \geq 3 \), then every voting system is either a dictatorship or is susceptible to manipulation.

Ouch. On Friday, we’ll prove these theorems. (For the source of the proof of Arrow’s theorem, see the link on the course webpage.)