

Jeremy L. Martin

Summary of Research Program

My research concerns combinatorics—enumerative, algebraic, geometric and topological—and its applications to diverse areas of mathematics. I have focused especially on algebraic moduli spaces such as graph varieties, much of whose structure can be described using combinatorial tools, including rigidity theory, matroids, Stanley-Reisner complexes, and the Tutte polynomial. Partly as a consequence of this work, I have started to explore manifestations of matroid theory beyond its combinatorial origins. Conversely, in my work on purely combinatorial problems, I often use tools from other fields, including Gröbner basis theory and the geometry of ℓ_p -spaces.

1. GRAPH VARIETIES AND COMBINATORIAL RIGIDITY THEORY

A main thrust of my work [Mar03, Mar04, Mar05] has been developing the theory of *graph varieties*, algebraic moduli spaces whose points parametrize geometric realizations of graphs. Some graph varieties resemble Hilbert schemes of points in space, while others are more closely related to the Fulton–MacPherson *compactification of configuration space* [FM94] and the De Concini–Procesi *wonderful model of subspace arrangements* [DCP95]. In addition to graph theory, combinatorial ideas such as the Tutte polynomial and rigidity theory play significant roles in describing the geometry and topology of graph varieties.

A *picture* of a graph G is a collection of points and lines representing the vertices V and edges E . The points and lines lie in some ambient space of dimension d (typically projective space over an algebraically closed field), and satisfy incidence conditions given by G . The term “graph variety” encompasses three algebraic sets constructed from G using pictures. The set $\mathcal{X}^d(G)$ of all pictures is called the *d -dimensional picture space* of G . The Zariski closure of the *generic* pictures—those for which the points are all distinct—is a component of $\mathcal{X}^d(G)$ of dimension $d|V|$, called the *picture variety*. Finally, forgetting all data except the direction vectors (in suitable coordinates) of the lines of a picture yields the *slope variety* $\mathcal{S}^d(G)$.

1.1. Rigidity Theory. Consider a physical model of G , constructed from joints representing the vertices V and bars representing the edges E , with the bars allowed either to vary in length (“telescope”) or to pivot about the joints. Whether such a model is rigid or flexible is a question of both theoretical and practical interest; comprehensive surveys include [GSS93] and [Whi96]. A vital fact is that the minimal rigid subgraphs of G form the bases of a *matroid*—a combinatorial abstraction of a vector space, in which such concepts such as linear independence, rank and span can be defined axiomatically. Allowing the bars to pivot but not to telescope yields the *d -rigidity matroid* $\mathcal{R}^d(G)$; allowing them to telescope but not to pivot produces the *d -parallel matroid* $\mathcal{P}^d(G)$.

Laman’s Theorem [Lam70] gives an elegant combinatorial description of the 2-rigidity matroid, which is known to coincide with the 2-parallel matroid [Whi96, Cor. 4.1.3]. For $d > 2$, the d -rigidity matroid remains rather mysterious; it does not coincide with the d -parallel matroid, and no generalization of Laman’s Theorem to higher dimensions is known.) Graph varieties provide an easy proof of the equality $\mathcal{R}^2(G) = \mathcal{P}^2(G)$: both 2-parallel independence and 2-rigidity independence are equivalent to the irreducibility of the picture space.

1.2. Defining Equations. The problem of determining the equations defining the picture variety $\mathcal{V}^2(G)$ can be rephrased in a very elementary way: given a set of distinct points in the plane, some pairs of which are connected with lines, determine the constraints on the slopes of those lines. A key to the solution is that each rigidity circuit (that is, minimal rigidity-dependent edge set) $C \subset E$ admits a decomposition into two edge-disjoint spanning trees [GSS93, §4.9]. Indeed, the

equations defining a picture of G imply the vanishing of a certain irreducible, homogeneous *tree polynomial* $\tau(C)$ on generic pictures of G . The monomials in $\tau(C)$ correspond to trees occurring in such decompositions. In fact, the picture variety is the vanishing locus of the tree polynomials in $\mathcal{X}^2(G)$, and that these same polynomials cut out $\mathcal{S}^2(G)$ set-theoretically [Mar03].

In the case that $G = K_n$, the complete graph on n vertices, the ideal generated by the tree polynomials has an unexpectedly rich structure [Mar04]. It is radical and Cohen-Macaulay, and has a Gröbner basis consisting of the tree polynomials of wheels (a particularly nice class of rigidity circuits). Moreover, its Hilbert series has a combinatorial description in terms of an elementary statistic on perfect matchings, first studied by Kreweras and Poupard in [KP78]. The proof of these facts makes extensive use of the theory of Stanley-Reisner rings and simplicial complexes.

1.3. The Topology of the Picture Space. The Tutte polynomial of a graph generalizes many other invariants, such as the number of spanning forests and the chromatic polynomial; for a comprehensive survey, see [BO92]. It can be defined recursively using deletion and contraction; these operations induce natural morphisms of the corresponding picture spaces, and in fact passing from G to $\mathcal{X}^d(G)$ is functorial (with respect to a suitable notion of a morphism of graphs). In [Mar05], I calculate the Poincaré series of the picture space (the generating function for the ranks of the homology groups $H_i(\mathcal{X}^d(G); \mathbb{Z})$) as a specialization of the Tutte polynomial. A consequence for rigidity theory is that the Tutte polynomial controls d -parallel independence, because whether or not the picture space is irreducible can be read off from its Poincaré series.

1.4. Questions for Further Study. Many questions about graph varieties remain unanswered. These fall into the following broad categories: extending the results of [Mar03] and [Mar04] to arbitrary ambient dimension; further investigating the combinatorial commutative algebra of tree polynomials; exploring the connections between graph varieties and other moduli spaces such as the De Concini–Procesi wonderful model and the Hilbert scheme of points in the plane; and using graph varieties as a means to study rigidity theory.

2. MATROID THEORY

2.1. Rigidity Theory for Matroids. The Tutte polynomial is really an invariant of *matroids*, not just of graphs, raising the question of whether it is possible to develop a rigidity theory for matroids that generalizes the existing theory of graph rigidity. As Develin, Reiner and I explain in [DMR04b], there are three ways to do so: combinatorially, linear-algebraically, and geometrically. When $d = 2$, the three constructions coincide, generalizing Laman’s Theorem to all representable matroids. More generally, many of the results of rigidity theory for graphs extend to the setting of matroids, with the advantage that the coordinate-free nature of matroids simplifies many of the arguments. Other geometric invariants, such as the cross-ratio of four lines in a plane, can be seen to interact with the combinatorics of matroids.

2.2. Cyclotomic and Simplicial Matroids. Let n be a natural number and ζ a primitive n th root of unity. It is well known that the degree of the cyclotomic field extension $\mathbb{Q}(\zeta)$ is $\phi(n)$, the Euler totient function. In [MR04], Reiner and I determine which sets of roots of unity form vector space bases for $\mathbb{Q}(\zeta)$. We prove that the matroids represented by the roots of unity are dual to the *simplicial matroids* studied by Lindström in [Lin81]. The bases themselves correspond to the “colorful simplicial trees” studied by Bolker [Bol76], Kalai [Kal83] and Adin [Adi92]. As an application of our result, when coupled with a technique of Ardila [Ard04], we obtain a generating function for the Tutte polynomials and related invariants of complete bipartite graphs.

2.3. Spanning Trees of the n -Cube. There is an elegant formula [Sta99, Example 5.6.10] for the number of spanning trees of Q_n , the 1-skeleton of the n -dimensional cube. No bijective proof has been found; the only known proof uses Kirchhoff's Matrix-Tree Theorem. In [MR03], Reiner and I weight each tree with a monomial that records the number of edges of T that lie in each direction and in each maximal face of Q_n , and prove that the corresponding generating function splits into quadratic factors with a simple combinatorial description. Our theorem may indicate the general form of a bijection that counts the spanning trees of Q_n . The methods of [MR03] are applicable to many classes of graphs, including threshold graphs (a special case of a result of Remmel and Williamson [RW02]) and arbitrary products of complete graphs.

2.4. Weighted Enumeration of Matroid Bases. The linear-algebraic techniques used to study the n -cube and other graphs in [MR03] can be extended to matroids. Indeed, the bases of an arbitrary representable matroid can be enumerated using a generalized Matrix-Tree Theorem, in which each basis is weighted by the square of the rank of a certain torsion group [Kal83]. In light of the combinatorial structure of tree polynomials and the matroid rigidity theory discussed above, it is plausible that such generating functions may encode the geometry of the matroid or its various rigidity matroids.

3. OTHER PROBLEMS IN COMBINATORICS AND GRAPH THEORY

3.1. Cohomology of Schubert Varieties. The cohomology rings of Schubert varieties have well-known presentations in terms of combinatorial objects such as symmetric functions and tableaux. In [DMR04a], Develin, Reiner and I study the cohomology rings of type A Schubert varieties X corresponding to dominant permutations. As observed by Ding [Din97, Din01], each Schubert cell in X corresponds to a rook placement on a Ferrers diagram, and the dimension of each cell can be computed combinatorially, leading to a formula for the Poincaré polynomial of X . Our main theorem is that the varieties X are determined up to isomorphism by their cohomology rings, and that the isomorphism classes are naturally indexed by multisets of certain "indecomposable" partitions. We plan to extend our classification to more general families of Schubert varieties.

3.2. Random Geometric Graphs. Let $\lambda > 0$ be a real number and $p \in (1, \infty]$. The *unit disk random geometric graph* G is the graph whose vertices are points distributed uniformly and independently in the unit disk in \mathbb{R}^2 , with two vertices adjacent if and only if their ℓ_p -distance is at most λ . The study of this random graph is of both theoretical and practical interest. In [EMY04], Ellis, Yan and I study the behavior of G as $n \rightarrow \infty$. Building upon work of Penrose [Pen99], we show that the connectivity threshold for G is $\lambda_0 = (A_p/\pi)^{-1/2} \sqrt{\ln n/n}$, where A_p is the area of the ℓ_p -unit disk. That is, G is almost always disconnected if $\lambda < \lambda_0$, and almost always connected if $\lambda > \lambda_0$. In the disconnected case, we calculate the number of isolated vertices; in the connected case, we determine upper bounds for the diameter of G .

3.3. Lacings and the Fundamental Theorem of Algebra. An n -lacing consists of $4n$ distinct points on a circle, colored alternately red and black, together with a noncrossing matching on each color class. While noncrossing matchings have been studied extensively by modern combinatorialists (see, e.g., [Sim00]), lacings appear to have escaped notice. Lacings arise in Gauss's first proof of the Fundamental Theorem of Algebra [Gau99]: for each complex polynomial $f(z)$ of degree n , there is a lacing given by drawing the real and imaginary parts of the equation $f(z) = 0$ as black and red curves, and intersecting the curves with a sufficiently large circle about the origin. Savitt, Singer and I are currently studying the problem of enumerating n -lacings by the number of red-black crossings, as well as generalizing to an arbitrary number of colors.

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