Jeremy L. Martin
Research Statement

My research concerns combinatorics—enumerative, algebraic, geometric and topological—and its applications to diverse areas of mathematics. Often but not always, the objects of my study originate in geometry. I have focused especially on algebraic moduli spaces such as graph varieties and Schubert varieties, much of whose structure can be described using combinatorial tools, including simplicial complexes, rigidity theory, matroids, and the Tutte polynomial. Partly as a consequence of my work on graph varieties, I have started to explore manifestations of matroid theory beyond its combinatorial origins: for instance, in algebraic and discrete geometry and in number theory. Conversely, in my work on purely combinatorial problems, I often use tools from other fields, including Gröbner basis theory and the geometry of ℓ_p-spaces.

1. Graph Varieties

A main thrust of my work has been developing the theory of *graph varieties*, algebraic moduli spaces whose points parametrize geometric realizations of graphs. Some graph varieties resemble Hilbert schemes of points in space, while others are more closely related to the Fulton–MacPherson *compactification of configuration space* [FM94] and the De Concini–Procesi *wonderful model of subspace arrangements* [DCP95]. In addition to graph theory, combinatorial ideas such as the Tutte polynomial and rigidity theory play significant roles in describing the geometry and topology of graph varieties.

1.1. An Overview of Graph Varieties.

Let G be an undirected graph with vertices V and edges E. A *picture* $\mathbf{P}$ of G consists of a point $\mathbf{P}(v) \in \mathbb{P}^d$ for each vertex v and a line $\mathbf{P}(e) \subset \mathbb{P}^d$ for each edge e, such that $\mathbf{P}(v)$ lies on $\mathbf{P}(e)$ whenever the vertex v is an endpoint of the edge e. The points and lines lie in the ambient space of projective d-space $\mathbb{P}^d$ over a field $\mathbb{F}$. Thus a picture may be regarded as a point in a product of Grassmannians indexed by the vertices and edges. The term *graph variety* encompasses several algebraic sets associated with the graph G.

The *picture space* $\mathcal{X}^d(G)$ is defined as the set of all pictures of G. It admits a decomposition into locally closed subvarieties called *cellules*, of which the most important is the *discrete cellule*, the set of all pictures $\mathbf{P}$ for which the points $\mathbf{P}(v)$ are distinct. The Zariski closure of the discrete cellule is called the *picture variety* $\mathcal{V}^d(G)$, an irreducible component of $\mathcal{X}^d(G)$ of dimension $2|V|$. (The other components correspond to families of "degenerate" pictures in which several points or lines coincide.) The *slope variety* $\mathcal{S}^d(G)$ is obtained by projecting an affine open patch of the picture variety onto coordinates giving the slopes (or direction vectors) of the lines $\mathbf{P}(e)$.

The component structure of $\mathcal{X}^d(G)$ resembles that of the Hilbert scheme of points in $\mathbb{F}^d$ (see, e.g., [MS04, §18.4]): a distinguished component corresponding to ideals of distinct points, and several other "junk" components. A fundamental difference between graph varieties and Hilbert schemes is that the points of a picture are labeled by the vertices of G, so that the picture space (for instance) is not obtained as a quotient by the action of the symmetric group. The Fulton–MacPherson and De Concini–Procesi spaces arise as desingularizations of the picture variety, the former when G is the complete graph, the latter in general.

1.2. Rigidity Theory.

Consider a physical model of G, constructed from joints representing the vertices V and bars representing the edges E, with the bars allowed either to vary in length ("telescope") or to pivot about the joints. Whether such a model is rigid or flexible is a question of both theoretical and practical interest; comprehensive surveys include [GSS93] and [Whi96]. A key observation is that the minimal rigid subgraphs of G are the bases of a *matroid* on E. (A matroid is a combinatorial abstraction of a vector space, in which such concepts such as linear independence, rank and span can be defined axiomatically and without coordinates.) Allowing the bars to pivot but not to telescope yields the *d-rigidity matroid* $\mathcal{R}^d(G)$; allowing them to telescope but not to pivot produces the *d-parallel matroid* $\mathcal{P}^d(G)$.

Laman's Theorem [Lam70] gives an elegant combinatorial description of the 2-rigidity-independent edge sets $E' \subset E(G)$: it is necessary and sufficient that

$$|E''| \leq 2|V(E'')| - 3 \tag{1}$$

for all nonempty subsets $E'' \subseteq E$, where $V(E'')$ is the set of endpoints of elements of E''. This completely characterizes $\mathcal{R}^2(G)$, and this description coincides with that of $\mathcal{P}^2(G)$ [Whi96, Cor. 4.1.3]. (For $d > 2$, the d-rigidity matroid remains rather mysterious; it does not coincide with the d-parallel matroid, and no generalization of Laman's Theorem to higher dimensions is known.)

Graph varieties provide an easy proof of the equality $\mathcal{R}^2(G) = \mathcal{P}^2(G)$. The idea is that both 2-parallel independence and 2-rigidity independence are equivalent to simple geometric statements about the irreducibility of the picture space and the dimensions of its cellules.

Theorem 1. [Mar03] *For every graph G, the following are equivalent:*

1. *Every nongeneric cellule of G has dimension strictly less than $2|V(G)|$;*
2. *G is 2-rigidity independent;*
3. *G is 2-parallel independent;*
4. *$\mathcal{X}^2(G) = \mathcal{V}^2(G)$.*

Problem 1. Explore the relationship between combinatorial rigidity and the geometry of other algebraic moduli spaces.

In Section 2.1, I describe an ongoing project (joint with Develin and Reiner) to solve this problem. Part of our approach involves developing a matroidal analogue of the picture space, which enables us to generalize several fundamental results of rigidity theory (such as Laman's Theorem), place them in a larger context, and simplify their proofs.

Problem 2. For $d > 2$, what does the d-dimensional picture space of a graph G say about the combinatorial properties of its d-rigidity matroid?

1.3. **Defining Equations.** The problem of determining the equations defining the picture variety $\mathcal{V}^2(G)$ can be rephrased in a very elementary way: Given a set of distinct points in the plane, some pairs of which are connected with lines, determine the constraints on the slopes of those lines. A key ingredient of the solution is the following property [GSS93, §4.9] of the *rigidity circuits*, the minimal dependent sets of the 2-rigidity matroid. Each rigidity circuit $C \subset E$ admits a decomposition into two edge-disjoint spanning trees, and the constraints on $\mathcal{V}^2(G)$ can be described in terms of these decompositions.

Theorem 2. [Mar03] *Let G be a graph with edge set E. Fix local coordinates on the picture space $\mathcal{X}^2(G)$ in which the line $\mathbf{P}(e)$ has slope m_e. Then:*

1. *The picture variety $\mathcal{V}^2(G)$ is cut out in $\mathcal{X}^2(G)$ by equations $\tau(C) = 0$, where C ranges over all rigidity circuits of G and $\tau(C)$ is a polynomial in the m_e, given explicitly by a certain determinant. These same equations cut out $\mathcal{S}^2(G)$ in* $\operatorname{Spec} \mathbb{F}[m_e]$.
2. *For each rigidity circuit C, the polynomial $\tau(C)$ is irreducible, and homogeneous of degree*

$|V(C)|-1$. *Moreover, it can be written as a sum of squarefree monomials:*

$$\tau(C) = \sum_T \left(\pm \prod_{e \in T} m_e \right),$$

where the sum ranges over all spanning trees T of C such that $C-T$ is also a spanning tree.

Problem 3. For general $d \geq 2$, give a combinatorial characterization of the equations cutting out $\mathcal{V}^d(G)$ as a subvariety of $\mathcal{X}^d(G)$.

Problem 4. Use Theorem 2 to give a combinatorial interpretation of the Hilbert series of the coordinate ring of $\mathcal{S}^2(G)$, for an arbitrary graph G. Additionally, determine whether the slope variety is Cohen–Macaulay.

1.4. **The Slope Variety of the Complete Graph.** My paper [Mar04] solves Problem 4 completely in the case that $G = K_n$ is the complete graph on vertices $V = \{1, 2, \ldots, n\}$. Gröbner bases, Stanley–Reisner complexes, matchings and labeled plane trees all play significant roles in the solution, as do a special class of rigidity circuits called *wheels*.

Theorem 3. [Mar04] *Let $n \geq 3$ and $G = K_n$. Let $R = \mathbb{F}[m_{1,2}, \ldots, m_{n-1,n}]$, and let $I \subset R$ be the ideal generated by the tree polynomials of rigidity circuits in G. Then:*

1. $\sqrt{I} = I$. Therefore, $\mathcal{S}(K_n) = \operatorname{Spec} R/I$ as a reduced scheme.

2. The tree polynomials of wheels in K_n form a Gröbner basis for I with respect to a certain natural graded term order.

3. The Hilbert series of R/I is a rational function $k(t)/(1-t)^{2n-3}$, where $k(t)$ enumerates perfect matchings by the number of long pairs. In particular, $\mathcal{S}^2(K_n)$ has dimension $2n-3$ and degree $d_n = (2n-5)(2n-7)\cdots(3)(1)$.

4. $\mathcal{S}^2(K_n)$ is Cohen-Macaulay.

The proof focuses on the monomial ideal J generated by the squarefree initial terms of the polynomials $\tau(W)$, and its Stanley–Reisner simplicial complex [BH93, Sta96] of squarefree monomials that do not belong to J. This complex is shellable, hence Cohen–Macaulay, and has exactly d_n facets; moreover, its h-vector consists of the coefficients of $k(t)$. (The long-pair statistic on perfect matchings was studied previously in [KP78].) To prove that J is the full initial ideal of I, it suffices to show that the slope variety has degree at least d_n. This is achieved by intersecting $\mathcal{S}(K_n)$ with suitably chosen hyperplanes and describing the intersections recursively.

Problem 5. Use Theorem 3 to describe the Hilbert series of $\mathcal{V}^2(K_n)$.

Conjecture 1. The tree polynomials of wheels form a *universal* Gröbner basis for I.

Conjecture 2. For every graph G, the slope variety $\mathcal{S}^2(G)$ is Cohen–Macaulay.

These conjectures are supported by experimental evidence, but both appear quite difficult. Conjecture 1 is especially problematic, because the term order used in Theorem 3 respects the natural ordering of the vertices (that is, the variables of R are ordered $m_{12} > m_{13} > \cdots > m_{1n} > m_{23} > \ldots$), while arbitrary term orderings on R need not do so.

1.5. **The Topology of the Picture Space.** The Tutte polynomial of a graph generalizes many other invariants, such as the number of spanning forests and the chromatic polynomial. It can be defined recursively using the elementary graphic operations of *deletion* and *contraction*. For a comprehensive survey, see [BO92]. These operations also induce natural morphisms of the corresponding picture spaces, and in fact $\mathcal{X}^d$ is a functor from graphs to algebraic sets (with respect to a suitable notion of a morphism of graphs). Thus, it is natural that the Tutte polynomial contains information about the picture space. Specifically, the homology groups of $\mathcal{X}^d(G)$ are determined completely by the Tutte polynomial of G.

Theorem 4. [Mar05] *For every graph G and every $d \geq 2$, the picture space $\mathcal{X}^d(G)$ over $\mathbb{C}$ is path-connected and simply connected, and its homology groups $H_i(\mathcal{X}^d(G);\mathbb{Z})$ are free abelian for i even and zero for i odd. Moreover, its Poincaré series—the generating function for the even Betti numbers* $\operatorname{rank}_{\mathbb{Z}} H_i(\mathcal{X}^d(G);\mathbb{Z})$*—is given by a specialization of the Tutte polynomial of G.*

Problem 6. Determine whether analogous topological properties hold for picture spaces over other fields, such as $\mathbb{R}$.

The number of components of $\mathcal{X}^d(G)$ of maximum dimension can be read off from the Poincaré series. The picture variety is always a component of minimum dimension, so the irreducibility of the picture space—and, by Theorem 1, rigidity properties of G—are controlled by its Tutte polynomial. Thus we have another link between combinatorial rigidity and graph varieties.

2. Matroid Theory

I continue to look for ways to use the ideas of matroid theory to problems arising in other areas of mathematics. In combinatorial rigidity, a purely matroidal approach promises to strengthen many of the fundamental results of the subject and to simplify their proofs. In another direction, matroid theory helps to explain a connection between cyclotomic fields and certain simplicial complexes generalizing complete bipartite graphs.

2.1. Rigidity and Matroids.

By Theorem 4, the Tutte polynomial of a graph controls its d-parallel independence or dependence. The Tutte polynomial is an isomorphism invariant of matroids as well as of graphs, raising the question of whether a theory of rigidity can be developed for a more general class of matroids than the graphic ones. That is, if $d \geq 2$ is an integer, is there a sensible way to define the "d-dimensional rigidity matroid" of a given matroid M? In fact, as Develin, Reiner and I describe in [DMR04b], there are *three* sensible ways:

(1) *Combinatorially*, by rewriting Laman's criterion (1) in purely matroidal terms;
(2) *Linear-algebraically*, by generalizing the construction [GSS93, p. 21] of the matrix representing the rigidity matroid of a graph;
(3) *Geometrically*, by an analogue of Theorem 1 using the *photo space* of M, a matroidal analogue of the picture space.

The combinatorial and linear-algebraic constructions always yield matroids (on the ground set of M), while the geometric construction produces a simplicial complex that may not be a matroid. Also, the combinatorial construction is well-defined for any matroid, while the other two require that M be representable over some field—a much less restrictive condition than that it be graphic.

When $d = 2$, the three constructions coincide, giving a "generalized Laman's Theorem" for representable matroids. We can explain other geometric phenomena: for instance, the matroid represented by four lines in 2-space is never rigidity-independent, suggesting that there exists an invariant governing the behavior of the lines—in this case, the cross ratio. We continue to explore other ways in which these rigidity matroids of matroids arise naturally in geometry.

2.2. Cyclotomic and Simplicial Matroids.

Let n be a natural number, let ζ be a primitive nth root of unity, and let $Z_n = \{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}$. It is well known that the degree of the cyclotomic field extension $\mathbb{Q}(\zeta)$ is $\phi(n)$, the Euler totient function. This raises the problem of determining which subsets of Z_n of cardinality $\phi(n)$ form vector space bases for $\mathbb{Q}(\zeta)$.

Theorem 5. [MR] *Let $p_1^{m_1} \dots p_r^{m_r}$ be the prime factorization of n, let Δ be the simplicial join of r point sets of cardinalities $p_1, \dots, p_r$, and let S be the corresponding simplicial matroid* [Lin81]. *Then the matroid represented by Z_n is dual to the direct sum of $p_1^{m_1-1} \dots p_r^{m_r-1}$ copies of S.*

The bases of S are the "colorful simplicial trees" studied by Bolker [Bol76], Kalai [Kal83] and Adin [Adi92].

Our result implies an upper bound for the number of subsets of Z_n that form bases for $\mathbb{Q}(\zeta)$. The bound is sharp if and only if n has at most two odd prime factors. For example, when p, q are distinct primes and $n = pq$, there is an explicit bijection between the bases of μ_n and the spanning trees of the complete bipartite graph $K_{p,q}$. In this case, the Tutte polynomial of the cyclotomic matroid can be computed by a method of Ardila [Ard04].

3. Cohomology of Grassmannian, Flag, and Schubert Varieties

The study of Grassmannian, flag, and Schubert varieties lies at the junction of combinatorics, algebraic geometry and representation theory. The cohomology rings of these spaces have well-known presentations in terms of combinatorial objects such as symmetric functions and tableaux. I have focused on extracting information from the cohomology rings using Gröbner basis theory.

3.1. **Classifying Rook Schubert Varieties.**

Let X be a type A Schubert variety X corresponding to a dominant permutation. As observed by Ding [Din97, Din01], each Schubert cell in X corresponds to a rook placement on a Ferrers diagram, and the dimension of each cell can be computed combinatorially, leading to a formula for the Poincaré polynomial of X. Develin, Reiner and I study the cohomology rings of these Schubert varieties in [DMR04a] using Gröbner basis theory. We prove that the varieties X are classified up to isomorphism by their cohomology rings, and that the isomorphism classes have a natural combinatorial description as multisets of certain "indecomposable" partitions.

Problem 7. Extend this classification to all (type A) Schubert varieties.

A starting point for such an extension of our results is the presentation of the cohomology rings of smooth Schubert varieties given by Gasharov and Reiner in [GR02]. One approach to Problem 7 might be to develop an analogous presentation for arbitrary type A Schubert varieties, then employ Gröbner basis methods as in [DMR04a].

3.2. **The Cohomology of the Grassmannian.**

Let k and ℓ be nonnegative integers, $n = k + \ell$, and let G be the Grassmannian variety of k-dimensional subspaces of $\mathbb{C}^n$. Classical Schubert calculus [Ful97, chapters. 9,10] presents the cohomology ring $R_{k,\ell} = H^*(G, \mathbb{Z})$ as an $\mathbb{N}$-graded algebra, generated by the elementary symmetric functions $e_1, \dots, e_k$, modulo the Jacobi–Trudi relations. One consequence is that the Hilbert series of $R_{k,\ell}$ is the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)\cdot(1-q)\cdots(1-q^\ell)}.$$

This polynomial has a well-known combinatorial interpretation as a generating function for partitions contained in a $k \times \ell$ rectangle.

For $1 \leq m \leq k$, Reiner and Tudose [RT] consider the subring $R_{k,\ell,m} \subset R_{k,\ell}$ generated by the first m elementary symmetric functions. They conjecture a formula for the Hilbert series of $R_{k,\ell,m}$; as with $R_{k,\ell}$, the formula has a combinatorial interpretation in terms of partitions. For small values of

k, ℓ and m, it is easy to compute the ideal $I_{k,\ell,m}$ defining $R_{k,\ell,m}$, and thus to verify the conjecture for particular cases, but no general proof has yet been found.

One approach to the Reiner–Tudose conjecture involves Gröbner basis theory. One can pass to an initial ideal J of $I_{k,\ell,m}$ and examine the "Stanley–Reisner multicomplex" Δ consisting of monomials not belonging to J. Since Δ determines the Hilbert series of $R_{k,\ell,m}$, the problem centers around finding a criterion for membership in Δ. In addition, one must choose an appropriate term order so that Δ has a combinatorially nice form. Experimentally, this method shows some promise, but the conjecture remains open.

4. Enumerative Combinatorics

I have studied a variety of problems in enumerative combinatorics, often but not always arising from graph theory. These include enumerating spanning trees of the n-dimensional cube; finding asymptotic bounds for the connectivity and diameter of random geometric graphs; and, most recently, investigating interlaced systems of noncrossing matchings, a project inspired by Gauss' first proof of the Fundamental Theorem of Algebra. Problems such as these require little technical background to understand, hence offer ideal opportunities for mentoring younger mathematicians. I have presented the n-cube problem to audiences of undergraduates, and one of my collaborators in the noncrossing-matching project is a high school student.

4.1. **Spanning Trees of the n-Cube.** For a connected graph G, let $c(G)$ denote the number of spanning trees of G (sometimes called the *complexity*). Certain classes of graphs have nice combinatorial formulas for $c(G)$. Perhaps the best known is Cayley's formula $c(K_n) = n^{n-2}$, which can be proved bijectively in various ways, such as the Prüfer code [Sta99, pp. 25–26]. In fact, the Prüfer code implies a more general statement:

$$\sum_T x_1^{d_T(1)} \cdots x_n^{d_T(n)} \;=\; x_1 \cdots x_n (x_1 + \cdots + x_n)^{n-2}. \tag{2}$$

the sum over all spanning trees of K_n, where $d_T(i)$ denotes the degree of vertex i in T. This formula can also be deduced from a weighted version of the Matrix-Tree Theorem.

The *n-cube* Q_n is the graph whose vertices are the binary words of length n, with two vertices are adjacent if they differ in exactly one place. The weighted Matrix-Tree Theorem can be used to prove the elegant formula [Sta99, Example 5.6.10]

$$c(Q_n) = \prod_{k=2}^{n} (2k)^{\binom{n}{k}}. \tag{3}$$

Problem 8. Prove the formula (3) bijectively.

This problem remains unsolved [Sta99, Sta]. In [MR03], Reiner and I prove a stronger version of the formula (3). We weight each spanning tree T of Q_n by a Laurent monomial $\mathrm{wt}(T)$ in variables $q_1, \ldots, q_n, x_1, \ldots, x_n$. The weight records the number of edges of T that lie in each direction, and in each maximal face of the n-cube.

Theorem 6. [MR03] *The generating function*

$$\sum_T \mathrm{wt}(T),$$

where T ranges over all spanning trees of Q_n, factors as the product of 2^{2^n-n-1} irreducible Laurent polynomials $f_S(q_1, \ldots, q_n, x_1, \ldots, x_n)$, indexed by the subsets $S \subseteq \{1, \ldots, n\}$ with $|S| \geq 2$.

In particular, $f_S(1, 1, \ldots, 1) = 2|S|$, which implies (3). The factorization of Theorem 6 may indicate the general form of a solution to Problem 8.

The proof of Theorem 6 uses the method of *identification of factors* [Kra99] and a weighted version of the Matrix-Tree Theorem. This technique yields analogous results for products of complete graphs and for threshold graphs (the latter case was considered in more generality in [RW02]).

The bases of an arbitrary representable matroid can be enumerated using a generalized Matrix-Tree Theorem, in which each basis is weighted by the square of the rank of a certain torsion group [Kal83]. This suggests the following problem:

Problem 9. Find factorizations of weighted enumerators of bases of other families of representable (not necessarily graphic) matroids.

4.2. **Random Geometric Graphs.** Let n be a positive integer, $\lambda > 0$ a real number, and $p \in (1, \infty]$. The *unit disk random geometric graph* G is the graph whose vertices are n points distributed uniformly and independently in the unit disk $D \subset \mathbb{R}^2$, with two vertices (x_1, y_1), (x_2, y_2) adjacent if and only if the ℓ_p-distance between them, namely $((x_1 - x_2)^p + (y_1 - y_2)^p)^{1/p}$, is at most λ. This random graph is of both theoretical and practical interest; for instance, it can be used to model a wireless communications network, as in [SSZ02].

In [EMY04], Ellis, Yan and I study the behavior of G as $n \to \infty$. Building upon work of Penrose [Pen99], we show that the connectivity threshold for G is

$$\lambda_0 = (A_p/\pi)^{-1/2}\sqrt{\ln n/n},$$

where A_p is the area of the ℓ_p-unit disk. That is, G is almost always disconnected if $\lambda < \lambda_0$, and almost always connected if $\lambda > \lambda_0$. In the former case, we find an asymptotic formula for the number of isolated vertices; in the latter case, we determine upper bounds for the diameter of G: an absolute bound K/λ independent of p (but involving a large constant K), and a bound depending on p that is tighter in most cases.

4.3. **Lacings and the Fundamental Theorem of Algebra.** A *noncrossing matching* of $2n$ points on a circle is a partition into pairs $\{a_1, b_1\}, \ldots, \{a_n, b_n\}$ such that for $i \neq j$, the segments $a_i b_i$ and $a_j b_j$ do not intersect. Noncrossing matchings, have been studied extensively by modern combinatorialists; see, e.g., the comprehensive survey [Sim00]. Savitt, Singer and I are presently studying generalizations of noncrossing matchings called *lacings*. David Savitt is a number theorist and a fellow counselor and teacher at Canada/USA Mathcamp; Theodore Singer is a high school student who began this project under Savitt's supervision at Mathcamp 2004.

An *n-lacing* consists of $4n$ points on a circle, colored alternately red and black, together with a noncrossing matching on each color class. (Thus the number of n-lacings is the square of the nth Catalan number.) For each complex polynomial $f(z)$ of degree n, there is a corresponding n-lacing given by drawing the real and imaginary parts of the equation $f(z) = 0$ as black and red curves, and intersecting the curves with a sufficiently large circle about the origin. Gauss observed that each red pair must cross at least one black pair, and based a proof of the Fundamental Theorem of Algebra on this observation.

Problem 10. Enumerate n-lacings by the number of red-black crossings. Generalize the answer to k-colored lacings on $2kn$ points.

Savitt, Singer and I are currently studying this problem, both for its pure combinatorial interest and with a view towards studying the solutions of complex polynomial equations.

References

[Adi92] R. Adin, *Counting colorful multi-dimensional trees*, Combinatorica **12** (1992), 247–260.

[Ard04] F. Ardila, *Computing the Tutte polynomial of a hyperplane arrangement*, `arXiv:math.CO/0409211`, 2004.

[BH93] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Univ. Press, Cambridge, 1993.

[BO92] T. Brylawski and J. Oxley, *The Tutte polynomial and its applications*, pp. 123–225, Cambridge Univ. Press, Cambridge, 1992.

[Bol76] E.D. Bolker, *Simplicial geometry and transportation polytopes*, Trans. Amer. Math. Soc. **217** (1976), 121–142.

[DCP95] C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), no. 3, 459–494.

[Din97] K. Ding, *Rook placements and cellular decompositions of partition varieties*, Discrete Math. **170** (1997), 107–151.

[Din01] ———, *Rook placements and classification of partition varieties* $B\backslash M_\lambda$, Commun. Contemp. Math. **3** (2001), 495–500.

[DMR04a] M. Develin, J.L. Martin, and V. Reiner, *Classification of Ding's Schubert varieties: finer rook equivalence*, Canadian J. Math., to appear, 2004, 2004.

[DMR04b] ———, *Rigidity theory for matroids*, in preparation, 2004.

[EMY04] R.B. Ellis, J.L. Martin, and C. Yan, *Random geometric graph diameter in the unit disk with* ℓ_p *metric*, Proceedings of the 12th International Symposium on Graph Drawing (New York, 2004), to appear, 2004.

[FM94] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) **139** (1994), no. 1, 183–225.

[Ful97] W. Fulton, *Young Tableaux*, Cambridge Univ. Press, New York, 1997.

[GR02] V. Gasharov and V. Reiner, *Cohomology of smooth Schubert varieties in partial flag manifolds*, J. London Math. Soc. (2) **66** (2002), no. 3, 550–562.

[GSS93] J. Graver, B. Servatius, and H. Servatius, *Combinatorial Rigidity*, American Mathematical Society, Providence, RI, 1993.

[Kal83] G. Kalai, *Enumeration of* $\mathbb{Q}$*-acyclic simplicial complexes*, Israel J. Math **45** (1983), 337–351.

[KP78] G. Kreweras and Y. Poupard, *Sur les partitions en paires d'un ensemble fini totalement ordonné*, Pub. Inst. Stat. Univ. Paris **XXIII** (1978), no. 1–2, 57–74.

[Kra99] C. Krattenthaler, *Advanced determinant calculus*, Sém. Lothar. Combin. **42** (1999), Art. B42q, 67 pp. (electronic), The Andrews Festschrift (Maratea, 1998).

[Lam70] G. Laman, *On graphs and rigidity of plane skeletal structures*, J. Engrg. Math **4** (1970), 331–340.

[Lin81] B. Lindström, *Matroids on the bases of simple matroids*, European J. Combin. **2** (1981), no. 1, 61–63.

[Mar03] J.L. Martin, *Geometry of graph varieties*, Trans. Amer. Math. Soc. **355** (2003), no. 10, 4151–4169.

[Mar04] ———, *The slopes determined by n points in the plane*, `arXiv:math.AG/0302106`, 2004.

[Mar05] ———, *On the topology of graph picture spaces*, Adv. Math. **191** (2005), no. 2, 312–338.

[MR] J.L. Martin and V. Reiner, *Cyclotomic and simplicial matroids*, Israel J. Math., to appear, 2004.

[MR03] ———, *Factorizations of some weighted spanning tree enumerators*, J. Combin. Theory Ser. A **104** (2003), no. 2, 287–300.

[MS04] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Studies in Mathematics, Springer-Verlag, New York, 2004, to appear.

[Pen99] M.D. Penrose, *On k-connectivity for a geometric random graph*, Random Structures Algorithms **15** (1999), 145–164.

[RT] V. Reiner and G. Tudose, *Conjectures on the cohomology of the Grassmannian*, `arXiv:math.CO/0309281`.

[RW02] J. Remmel and S.G. Williamson, *Spanning trees and function classes*, Electron. J. Combin. **9** (2002), Research Paper 34, 24 pp. (electronic).

[Sim00] R. Simion, *Noncrossing partitions*, Discrete Math. **217** (2000), no. 1-3, 367–409, Formal power series and algebraic combinatorics (Vienna, 1997).

[SSZ02] I. Stojmenovic, M. Seddigh, and J. Zunic, *Dominating sets and neighbor elimination-based broadcasting algorithms in wireless networks*, IEEE Trans. Parallel Distrib. Syst. **13** (2002), no. 1, 14–25.

[Sta] R.P. Stanley, personal communication, July 2004.

[Sta96] ———, *Combinatorics and Commutative Algebra*, 2nd ed., Birkhäuser, Boston, 1996.

[Sta99] ———, *Enumerative Combinatorics*, vol. 2, Cambridge Univ. Press, Cambridge, 1999.

[Whi96] W. Whiteley, *Some matroids from discrete applied geometry*, Contemp. Math. **197** (1996), 171–311.