Problem 2004-A1: Basketball star Shanille O'Keal’s team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

Solution: If not, then there must have been some free throw after which Shanille’s percentage went from below 80% to over 80%; that is, some value of $N$ for which

\[
\frac{S(N)}{N} < \frac{4}{5} \quad \text{but} \quad \frac{S(N + 1)}{N + 1} > \frac{4}{5}.
\]

Let’s abbreviate $s = S(N)$. Clearly Shanille must have made that $(N + 1)^{th}$ free throw, so $S(N + 1) = s + 1$ and we can rewrite these inequalities as

\[
\frac{s}{N} < \frac{4}{5} < \frac{s + 1}{N + 1}.
\]

But if $\frac{s}{N} < \frac{4}{5}$, then $5s < 4N$, so

\[
5(s + 1) = 5s + 5 < 4N + 5,
\]

and since $5(s + 1)$ and $4N + 5$ are integers, it must be the case that

\[
5(s + 1) \leq 4N + 4 = 4(N + 1),
\]

that is,

\[
\frac{s + 1}{N + 1}(s + 1) \leq \frac{5}{4},
\]

and this contradicts the second part of (1a). So (perhaps surprisingly) the answer is yes.

\[\square\]

Lesson: Try to rewrite inequalities in terms of integers when possible. If $a, b$ are integers, then $a < b$ can be replaced with $a \leq b - 1$. 

1
Problem 2004-A3: Define a sequence \( \{u_n\}_{n=0}^{\infty} \) by \( u_0 = u_1 = u_2 = 1 \), and thereafter by the condition that

\[
\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!
\]

for all \( n \geq 0 \). Show that \( u_n \) is an integer for all \( n \). (By convention, \( 0! = 1 \).)

Solution: The definition of the sequence can be rewritten as

\[ u_n u_{n+3} - u_{n+1} u_{n+2} = n! \]

or

\[ u_{n+3} = \frac{n! + u_{n+1} u_{n+2}}{u_n} \]

or

\[ u_n = \frac{(n-3)! + u_{n-1} u_{n-2}}{u_{n-3}}. \]

Let’s calculate the first few values to see what the sequence looks like:

\[
\begin{align*}
    u_0 &= 1, \\
    u_1 &= 1, \\
    u_2 &= 1, \\
    u_3 &= \frac{0! + 1}{1} = 2, \\
    u_4 &= \frac{1! + 1}{1} = 3, \\
    u_5 &= \frac{2! + 3}{2} = 8, \\
    u_6 &= \frac{3! + 8}{3} = 15, \\
    u_7 &= \frac{4! + 15}{4} = 48, \\
    u_8 &= \frac{5! + 48}{5} = 105, \\
\end{align*}
\]

The pattern is that

\[
\begin{align*}
    u_1 &= 1, \\
    u_2 &= 1, \\
    u_3 &= 2 = 2 \cdot 1, \\
    u_4 &= 3 = 3 \cdot 1, \\
    u_5 &= 8 = 4 \cdot 2, \\
    u_6 &= 15 = 5 \cdot 3, \\
    u_7 &= 48 = 6 \cdot 8, \\
    u_8 &= 105 = 7 \cdot 15, \\
\end{align*}
\]

It sure looks like \( u_n = (n-1)u_{n-2} \) for \( n \geq 3 \). If we can prove this, then it will certainly imply that \( u_n \) is an integer for all \( n \).

Maybe you can prove this; I couldn’t figure out how. However, here’s another way to describe the pattern:

\[
\begin{align*}
    u_1 &= 1, \\
    u_2 &= 1, \\
    u_3 &= 2, \\
    u_4 &= 3 \cdot 1, \\
    u_5 &= 4 \cdot 2, \\
    u_6 &= 5 \cdot 3 \cdot 1, \\
    u_7 &= 6 \cdot 4 \cdot 2, \\
    u_8 &= 7 \cdot 5 \cdot 3 \cdot 1, \\
\end{align*}
\]

It looks like

\[ u_n u_{n-1} = (n-1)! \]
for all $n$. This might be easier to prove. We’ll certainly have to use induction. The base case is no problem; the explicit calculations above show that it works for $n \leq 8$.

For the inductive step, suppose that $u_m u_{m-1} = (m-1)!$ for all $m < n$. Then

$$ u_n = \frac{(n-3)! + u_{n-1}u_{n-2}}{u_{n-3}} $$

$$ = \frac{((n-3)! + (n-2)!) u_{n-2}}{(n-3)!} $$

$$ = (1 + (n-1)) u_{n-2} $$

$$ = (n-1) \frac{(n-2)!}{u_{n-1}} $$

$$ = \frac{(n-1)!}{u_{n-1}} $$

which is equivalent to (2a), as desired.

**Lesson:** Be willing to do some calculation to look for a pattern. When you do look for the pattern, be flexible in what you try and prove.
Problem 2004-A5: An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability 1/2. We say that two squares, $p$ and $q$, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at $p$ and ending at $q$, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Solution: First of all, we should regard this setup as a graph $G$ with $mn$ vertices (the squares), and edges given by the pairs of adjacent squares of the same color, so that the “connected monochromatic regions” are the connected components of the graph.

Here’s a first try that didn’t work. Let $C$ be the number of components, and let $C_d$ be the number of components of size $d$. Then $C_1$ is the number of squares all of whose neighbors have the opposite color. (These correspond to isolated vertices in $G$.) We’ll calculate the expected value $\mathbb{E}(C_1)$, which is certainly less than $\mathbb{E}(C) = \mathbb{E}(C_1) + \mathbb{E}(C_2) + \ldots$. The calculation uses linearity of expectation:

$$\mathbb{E}(C_1) = \sum_{\text{squares } p} \mathbb{P}[p \text{ is a singleton component}].$$

That probability is just 1/2 to the number of neighbors of $p$, i.e.,

$$\mathbb{P}[p \text{ is a singleton component}] = \begin{cases} \frac{1}{4} & \text{if } p \text{ is a corner (4 cases)}, \\ \frac{1}{8} & \text{if } p \text{ is a side (2m - 2n - 8 cases)}, \\ \frac{1}{16} & \text{if } p \text{ is a middle ((m - 2)(n - 2) cases).} \end{cases}$$

Therefore

$$\mathbb{E}(C_1) = \frac{1}{4}(4) + \frac{1}{8}(2m - 2n - 8) + \frac{1}{16}(m - 2)(n - 2)$$

$$= \frac{mn}{16} + \frac{m + n}{8} + \frac{1}{4},$$

which is not quite good enough (we’ve only shown that $\mathbb{E}(C) > mn/16$, not $mn/8$). At least the calculation wasn’t hard.

Back to the drawing board. Can we try to calculate $\mathbb{E}(C_d)$ for some value of $d$ other than 1? That looks pretty ugly, as there are lots more possibilities for what a bigger component might look like, so the case-by-case calculation is likely to be intractably unwieldy.

What about trying to use some graph theory? If a graph has $V$ vertices, $A$ edges, and $C$ components, then

\[(3a) \quad C \geq V - A\]
The trick is to realize that a lot of these edges may be redundant for the purposes of calculating the expected number of components. This graph may have a bunch of cycles of length 4, formed by $2 \times 2$ blocks of squares with the same color, and whenever we have such a block, we can delete one of its edges without making any new components. This is true even if we have a block that is larger than $2 \times 2$, as long as we always delete the bottom edge.

![Graph](image.png)

What this amounts to is the statement that for this graph, $C \geq V - A + Q$, where $Q$ is the number of 4-cycles. By linearity of expectation,

\[(3b) \quad \mathbb{E}(C) \geq \mathbb{E}(V) - \mathbb{E}(A) + \mathbb{E}(Q) \geq \frac{m+n}{2} + \mathbb{E}(Q).\]

On the other hand, there are $(m - 1)(n - 1)$ potential 4-cycles, each of which will occur with probability $1/8$. Yet again by linearity of expectation, it follows that

\[(3c) \quad \mathbb{E}(Q) = \frac{(m - 1)(n - 1)}{8}\]

and combining this with (3b) gives us

\[(3d) \quad \mathbb{E}(C) \geq \frac{m+n}{2} + \frac{(m - 1)(n - 1)}{8} = \frac{mn + 3m + 3n + 1}{8} > \frac{mn}{8}\]

as desired. \hfill \Box

**Lesson:** Linearity of expectation is a trick that comes up all the time on the Putnam. Knowing a little graph theory (such as the formula (3a)) can be useful too, particularly if you can translate an exam problem into graph-theoretic terms.
Problem 2004-B4: Let $n$ be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the $xy$-plane, for $k = 1, 2, \ldots, n$. Let $R_k$ be the map that rotates the plane counterclockwise by the angle $\theta$ about the point $P_k$. Let $R$ denote the map obtained by applying, in order, $R_1$, then $R_2$, \ldots, then $R_n$. For an arbitrary point $(x, y)$, find, and simplify, the coordinates of $R(x, y)$.

Solution: The first thing is to realize that $R$ has to be a translation. Since rotation is an isometry (it preserves distances), $R$ must be as well. Let’s regard the plane as being the piece of paper you are holding. Each time you apply the transformation $R_i$, the angle of the text rotates by $\theta = 2\pi/n$, so that after you finish applying the composite $R = R_n \circ R_{n-1} \circ \cdots \circ R_1$, the text will be right-side up again. It just won’t be in the same place as it used to be, which means that we have to figure out which translation $R$ is.

It would suffice to calculate what $R$ does to any particular point. But doing this for an arbitrary point is going to be a nightmare, so maybe we can cleverly find a point $Q_0$ for which $R(Q_0)$ is easy to calculate.

How about a point $Q_0 = (a, b)$ whose $y$-coordinate is unchanged by $R_1$? Such a point would have to have the property that the line $L$ through $Q_0$ and $P_1$ makes an angle of $\pi/n = \theta/2$ with the vertical line $V$ through $P$, so that $R_1$ reflects $Q_0$ across $V$. That is,

$$Q_1 = R_1(Q_0) = R_1(a, b) = (2 - a, b).$$

If we choose $Q_0 = (x, y)$ to be the unique point on $\ell$ with $x = \frac{1}{2}$, then $Q_1 = R_1(Q_0) = (3/2, y)$. But then each successive rotation moves the resulting point an additional 1 unit to the right. That is, $R(Q_0) = (x + n, y)$. 

[Diagram of points $P_1$, $Q_0$, and $Q_1$ with lines and angles labeled.]
Since $R$ is a translation, it must be the case that $R(x, y) = (x + n, y)$ for all $(x, y)$.

**Lesson:** Look for shortcuts. In particular, avoid approaches that require lots of disgusting calculations (e.g., in this problem, trying to calculate $R(x, y)$ for general $x, y, n$) and try to come up with an easier way of finding out what you need (in this case, finding the point $Q_0$).
Problem 2004-B2: Let \( m \) and \( n \) be positive integers. Show that
\[
\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}
\]

Solution: The desired inequality (5a) is equivalent to the following one:
\[
\binom{m+n}{m} \cdot m^m \cdot n^n < (m+n)^{m+n}
\]

The expression \( \binom{m+n}{m} \) is the binomial coefficient \( \binom{m+n}{m} \), the number of choosing a subset of \( m \) objects from a set of \( m+n \) objects. Meanwhile, \( m^m \) (respectively, \( n^n \) or \( (m+n)^{m+n} \)) is the number of functions from a set of cardinality \( m \) (respectively, \( n \) or \( (m+n)^{m+n} \)) to another set with the same cardinality. This suggests that we should prove the inequality by trying to interpret the left side of (5b) combinatorially.

Let \( S = \{1, 2, \ldots, m+n\} \), and consider the set of functions
\[
X := \{ f : S \to S \mid |f^{-1}\{1, \ldots, m\}| = m \},
\]
that is, the number of functions \( f : S \to S \) such that \( f(x) \in \{1, \ldots, m\} \) for exactly \( m \) values of \( x \) (and therefore \( f(x) \in \{m+1, \ldots, m+n\} \) for the other \( n \) possible values of \( x \)).

I claim that \( |X| \) is exactly the left-hand side of (5b). Indeed, if I want to specify a member of \( X \), I can first choose a subset \( T \subset S \) of cardinality \( m \), then choose a function \( T \to \{1, \ldots, m\} \), then choose a function \( S - T \to \{m+1, \ldots, m+n\} \). On the other hand, \( X \) is a subset of the set of all functions \( S \to S \), and it is a proper subset because both \( m, n \) are nonzero (for instance, the constant functions do not belong to \( X \)). This proves the desired inequality.

Lesson: Look for binomial coefficients everywhere (they are everywhere!) Try to make sense of complicated expressions by interpreting them combinatorially.

This problem would have been easier if the inequality had been handed to you in the form (5b), rather than (5a). So, be on the lookout for ways to transform what you’re being asked to prove into a form that makes more mathematical sense. (Maybe the Putnam exam writers started out with the combinatorial inequality (5b) and transformed it into (5a) just to give you a hard time.)