Putnam Practice Problems I

Problem #1 (Easy) Take an 8 \times 8 checkerboard and remove two opposite corners. Is it possible to pair up the remaining 62 squares so that each square is paired with one of its neighbors? (The neighbors of a square are those squares with which it shares an edge.)

Problem #2 (Easy) Last I checked, McDonald’s was selling Chicken McNuggets in sizes of 6, 9, or 20. What’s the largest number of McNuggets that you can’t buy? Or is there any such number?

Problem #3 A confused secretary stuffs twelve letters into twelve separate envelopes and drops them in a mailbox. Then he realizes that he forgot to check that the right letters were in the right envelopes. (Oops.)

#3a (Easy) What’s the probability that every letter is in the right envelope?

#3b (Easy) What’s the probability that exactly eleven letters are in the right envelopes?

#3c (Medium) On average, how many letters will wind up in the right envelopes?

#3d (Medium) What’s the probability that every letter is in the wrong envelope?

Problem #4 (Medium) The (finitely many) dwarves of the Black Forest live in houses that are each painted either red or blue. Some dwarves are friends, and some aren’t. Every so often, a dwarf (let’s call him Grumpy) will think about repainting his house. To decide whether to repaint, Grumpy looks at the colors of his friends’ houses. If Grumpy’s house is red, but more of his friends live in blue houses than in red houses, then Grumpy will paint his own house blue. On the other hand, if Grumpy’s house is blue, but more of his friends live in red houses than in blue houses, then Grumpy will paint his own house red.

Prove that after some point in time, no dwarf ever paints his house again. (Yes, dwarves are immortal, and you may assume that the world contains an infinite supply of paint.)

Problem #5 (Medium) The Penny Game is played as follows. Two players take turns placing pennies on a circular table. Each penny must lie completely on the table, and cannot overlap any other penny. If it’s your turn and you have no legal moves, then you lose.

Do you want to go first or second? Why?

What about if the table was square instead of circular? Or triangular? Or octagonal?
Problem #6 (Medium) What is the probability of being dealt a pat full house at poker? (That is, three of one rank of card and two of another, such as three queens and two sixes.)

Problem #7 Twelve kids at summer camp need to use the buddy system when they go swimming: no one goes in the water without a buddy.

#7a (Easy) How many possible ways are there to pair the twelve kids into six pairs?

#7b (Medium) Wait a minute. Only eight of the twelve kids know how to swim, and clearly it wouldn’t be a good idea to pair two nonswimmers together—every nonswimmer is required to have a buddy who can swim. Now how many possibilities are there?

#7c (Medium) This time every one of the twelve kids (five boys and seven girls) knows how to swim, but there’s a different problem. All the boys think the girls have cooties, and vice versa. (Did I mention that these kids are all eight years old?) So every girl would rather have a girl for her buddy, and every boy would rather have a boy for his buddy. Of course, since there are an odd number of boys and an odd number of girls, someone is going to have to get cooties. Be that as it may, now how many possibilities are there?

Problem #8 (Hard) Four mathematicians (Kovalevskaia, Lagrange, Maclaurin and Noether) must cross a rickety bridge over a deep canyon. Kovalevskaia can cross the bridge by herself in 10 minutes, Lagrange can cross by himself in 5 minutes, Maclaurin can cross in 2 minutes, and Noether can cross in 1 minute. However, the bridge is so narrow that no more than two people can cross simultaneously. Also, it is dark and the bridge has a lot of gaps, so it can’t be crossed without a flashlight, and the group only has one flashlight.

Oh, I almost forgot. In precisely eighteen minutes, a flash flood will sweep away anyone who hasn’t crossed yet (and anyone who is still on the bridge). Can all four mathematicians be saved?

Problem #9 I’m sorting through a big pile of coins. I know that one of them is counterfeit, and I need to find out which one. The counterfeit coin looks just like the others, but it is lighter than each of the genuine coins (which all weigh the same). The only tool I have available is a balance scale, which is a pain to use (it works all right, but it’s old and rusty and squeaks horribly).

#9a (Easy) Suppose that there are n coins. What is the fewest number of weighings that I need to detect the counterfeit coin?

#9b (Hard) How about if I don’t know whether the counterfeit coin is lighter or heavier than the others, just that it has a different weight?
Problem #10 (Easy) I toss a fair coin 17 times, recording whether each toss comes up heads or tails. What is the probability that the coin comes up heads an even number of times? What is the probability that there are more heads than tails? Would the answers be the same if I tossed the coin 18 times instead?

Problem #11 (Easy) A lattice point in $\mathbb{R}^3$ is a point whose coordinates are all integers. Nine distinct lattice points are chosen at random and each pair of points is connected with a line segment. Show that at least one of these line segments contains a lattice point other than its endpoints.

Problem #12 (Hard) Let $n$ be a positive integer. A partition of $n$ is a sequence of positive integers $(a_1, a_2, \ldots, a_k)$ such that $a_1 \geq a_2 \geq \cdots \geq a_k$ and $a_1 + a_2 + \cdots + a_k = n$. A partition $(a_1, a_2, \ldots, a_k)$ is called odd if the numbers $a_i$ are all odd; it is called distinct if $a_1 > a_2 > \cdots > a_k$.

Prove that for every $n$, the number of odd partitions equals the number of distinct partitions.

Problem #13 (Medium) Let $A, B, C, D, E$ be five randomly chosen points on a sphere. Prove that there is some closed hemisphere (i.e., including the equator) that contains at least four of those points.

Problem #14 The Fibonacci numbers are the numbers $F_0, F_1, F_2, \ldots$ defined by

$$F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1}.$$  

So the Fibonacci sequence begins

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

#14a (Easy) For which $n$ is $F_n$ even?

#14b (Medium) Prove that every Fibonacci number that is divisible by 7 is divisible by 3.

#14c (Medium) Evaluate $\lim_{n \to \infty} \frac{F_n}{F_{n-1}}$.

#14d (Hard) Evaluate $\sum_{n=1}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}}$.

#14e (Hard) Evaluate $\sum_{n=1}^{\infty} \frac{1}{F_{n-1}F_{n+1}}$. 
Problem #15 (Medium) Evaluate\[ \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \cdots. \]

Problem #16 (Medium) Evaluate the sums \[ \sum_{k=0}^{n} k \binom{n}{k}. \]

Problem #17 (Hard) Evaluate \[ \sum_{k=0}^{n} k^2 \binom{n}{k}. \]

Problem #18 (Hard) Calculate \( \lim_{n \to \infty} \left( \binom{2n}{n} \right)^{1/n}. \)

Problem #19 (Hard) Find all real solutions to the system of equations\[
\begin{align*}
  x + y &= \sqrt{4z - 1}, \\
  x + z &= \sqrt{4y - 1}, \\
  y + z &= \sqrt{4x - 1}.
\end{align*}
\]

Problem #20 (Medium) For how many numbers \( n \) between 1 and 1023 is the binary expansion of \( n \) a palindrome?

Problem #21 (Very hard) Determine all pairs of distinct positive integers \( a, b \) such that \( a^b = b^a \).

Problem #22 (Medium) A basketball team plays 30 games in 20 days, playing at least one game every day. Show that there is some period of consecutive days during which the team plays exactly nine games.