Pseudodeterminants and perfect square spanning tree counts

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AMS Central Sectional Meeting
University of Wisconsin, Eau Claire
September 20, 2014

Cellular Trees

- $X$: pure cell complex (\(=\) CW complex) of dimension $d$
- $\partial_k$: cellular boundary map \(\partial_k : C_k(X) \to C_{k-1}(X)\)

**Tree in $X$:** $T = T_d \cup \text{Skel}_{d-1}(X)$ where $T_d = \text{column basis of} \ \partial_d$

- $H_d(T; \mathbb{Q}) = 0$
- $H_{d-1}(T; \mathbb{Q}) = H_{d-1}(X; \mathbb{Q})$

$\mathcal{T}_k(X) = \text{set of all} \ k\text{-trees in} \ X = \text{trees in} \ \text{Skel}_k(X)$

**Examples:**
- $\mathcal{T}_1(X) = \{\text{spanning forests of 1-skeleton graph}\}$
- $\mathcal{T}_0(X) = \{\text{individual vertices}\}$
- $\mathcal{T}_d(X \cong \mathbb{S}^d) = \{X - \sigma : \ \sigma \text{ a facet}\}$
Counting Cellular Trees

Assume $\tilde{H}_{k-1}(X; \mathbb{Q}) = 0$ (analogue of connectedness).

**Tree count:**

$$\tau_k(X) = \sum_{T \in \mathcal{T}_k(X)} |\tilde{H}_{k-1}(T; \mathbb{Z})|^2.$$

**Weighted tree count:** Assign each $\sigma \in X$ a monomial weight $q_{\sigma}$.

$$\tau_k(X; q) = \sum_{T \in \mathcal{T}_k(X)} |\tilde{H}_{k-1}(T; \mathbb{Z})|^2 \prod_{\sigma \in T} q_{\sigma}$$
Counting Cellular Trees

**Cellular matrix-tree theorem**: expresses $\tau_k(X), \tau_k(X;q)$ in terms of eigenvalues/cokernels of combinatorial Laplacians $\partial_k \partial_k^{tr}$.

- Bolker ’78: first studied simplicial spanning trees
- Kalai ’83: homology-squared weighting; skeletons of simplices
- Adin ’92: complete colorful complexes
- Duval–Klivans–JLM; Lyons; Catanzaro–Chernyak–Klein: general formulations

The cellular matrix-tree theorem can be restated in terms of **pseudodeterminants**.
Pseudodeterminants

The cellular matrix-tree theorem can be restated in terms of pseudodeterminants. What’s a pseudodeterminant?

Let $L \in \mathbb{Z}^{n \times n}$, not necessarily of full rank; eigenvalues $\lambda_1, \ldots, \lambda_n$.

**Pseudodeterminant $\text{pdet}(L)$**: last nonzero coefficient of characteristic polynomial $= \text{coefficient of } t^{n-\text{rank } L}$.

$$\text{pdet } L = \prod_{\lambda_i \neq 0} \lambda_i = \sum_{I \subseteq [n]: |I| = \text{rank } L} \det L_{I,I}$$

(So $\text{pdet } L = \det L$ if $L$ is of full rank.)
Counting Trees with Pseudodeterminants

**Cellular Matrix-Tree Theorem, Pseudodeterminant Version:**
Let $L_{ud}^k = \partial_k \partial_k^{tr}$, the $(k-1)^{th}$ updown Laplacian of $X$. (This is a linear operator on $C_{k-1}(X)$.) Then

$$\text{pdet } L_{ud}^k = \tau_k(X)\tau_{k-1}(X).$$

Classical matrix-tree theorem: $G$ graph, $L = L_{0}^{ud}(G)$.

$$\# \text{ spanning trees} = \frac{\text{product of nonzero eigenvalues of } L}{\text{number of vertices}}$$

$$\tau_1(G) = \text{pdet } L / \tau_0(G)$$
Pseudodeterminants and (Skew)-Symmetry

Proposition

Let $\partial \in \mathbb{Z}^{n \times n}$ be either symmetric or skew-symmetric. Then:

1. $\text{pdet}(\partial \partial^{\text{tr}}) = (\text{pdet} \partial)^2$.
2. All principal minors $\partial_{I,I}$ have the same sign, so

$$\text{pdet} \partial = \pm \sum_{I} |\text{coker} \partial_{I,I}| \quad \text{ (★)}$$

where $I$ ranges over all row bases of $\partial$.

Question

What topological setup will give (★) combinatorial meaning?
Perfect Square Phenomena in Spanning Tree Counts

Tutte: $G$ planar; $G \cong G^*$ from antipodal map on $S^2 \implies \tau(G) = (\text{number of self-dual spanning trees})^2$.

$\tau(W_3) = 16 = 4^2$

$\tau(W_5) = 121 = 11^2$

$\tau(W_7) = 841 = 29^2$

**Question**

Are there analogous perfect-square phenomena for higher-dimensional self-dual cell complexes?
Even-Dimensional Spheres: Maxwell’s Theorem

Theorem (Maxwell ’09)

Let \( k \) be odd. Let \( X \) be an antipodally self-dual cellular \( S^{2k} \) with at least one \( \mathbb{Z} \)-acyclic self-dual tree. Then

\[
\sum_{T \in \mathcal{T}_k(X)} |\tilde{H}_{k-1}(T; \mathbb{Z})|^2 = \left( \sum_{T \in \mathcal{T}_k(X), \text{T self-dual}} |\tilde{H}_{k-1}(T; \mathbb{Z})| \right)^2.
\]

What about odd-dimensional antipodally self-dual spheres?

- \( \dim = 2k \): involution on \( k \)-dimensional faces
- \( \dim = 2k + 1 \): pairing between \( k \) - and \( (k + 1) \)-dim’l faces
Self-Dual Cell Complexes

**Self-dual d-ball:** regular cell complex $B \cong \mathbb{B}^d$, with an anti-automorphism $\alpha$ of its face poset:

$$\sigma \subseteq \tau \iff \alpha(\sigma) \supseteq \alpha(\tau).$$

**Self-dual (d − 1)-sphere:** $S = \partial B \cong S^{d-1}$.

**Example:** $B =$ simplex on vertex set $V$; $\alpha(\sigma) = V \setminus \sigma$

**Example:** Self-dual polytopes (polygons in $\mathbb{R}^2$; pyramids over polygons in $\mathbb{R}^3$; the 24-cell in $\mathbb{R}^4$; . . . )
Proposition

Let $B$ be a self-dual cellular $\mathbb{B}^d$ and $j + k = d - 1$. Then $\tau_j(B) = \tau_k(B)$.

Proof sketch.
For $T \in \mathcal{T}_j(B)$, consider the Alexander dual

$$T^\vee = \{\sigma \in B : \alpha(\sigma) \notin T\}.$$

Then

$$\mathcal{T}_j(B) = \{T^\vee : T \in \mathcal{T}_k(B)\}$$

and

$$H_{j-1}(T; \mathbb{Z}) \cong H_{k-1}(T^\vee; \mathbb{Z}).$$
Perfect Square Phenomenon for Even-Dimensional Balls

Let $B \cong \mathbb{B}^{2k}$ be self-dual and let $\partial = \partial_k(B)$. Then

$$\tau_{k-1}(B) = \tau_k(B)$$

and the pseudodeterminant version of the CMTT says that

$$\text{pdet} (\partial \partial^{tr}) = \tau_{k-1}(X) \tau_k(X) = \tau_k(X)^2.$$ 

**Repeat Question:** What additional structure will enable

$$\text{pdet} \partial = \pm \sum_{i} |\text{coker } \partial_{i,i}| \quad (\star)$$

to carry combinatorial meaning?
**Definition:** A self-dual cellular $d$-ball $(B, \alpha)$ is **antipodally self-dual** if $\alpha$ arises from the antipodal map on $\partial B \cong S^{d-1}$.

Technical details: explicit orientations, dual block complex, Poincaré duality...
Antipodal Self-Duality and Orientations

**Proposition (Very Technical!)**

Let $B$ be an antipodally self-dual cellular $(2k)$-ball. Then $B$ can be oriented so that the middle boundary matrix $\partial_k$ satisfies

$$\partial^{tr} = (-1)^k \partial.$$ 

**Example**

If $B$ is the simplex on vertices $[2k + 1]$, then start with the “textbook” orientation and reorient:

$$\sigma = \{v_0, \ldots, v_k\} \in B_k \mapsto (-1) \sum v_i \sigma.$$
Antipodally Self-Dual Even-Dimensional Balls

Proposition

Let $B \cong \mathbb{B}^{2k}$ be antipodally self-dual. Then $B$ can be oriented so that the middle boundary matrix $\partial = \partial_k$ satisfies

$$\partial^{tr} = (-1)^k \partial.$$ 

Theorem

Let $B \cong \mathbb{B}^{2k}$ be antipodally self-dual and write $\tau_i = \tau_i(B)$. Then

$$\tau_k = \tau_{k-1} = \text{pdet} \partial \star \sum_I |\text{coker} \partial_{I,I}| = \sum_{T \in \mathcal{T}_k(S)} |H_k(T, T^\vee; \mathbb{Z})|.$$ 

(There is also a $q$-analogue.)
Open Questions

1. What about antipodally self-dual $\mathbb{B}^d$ with $d \equiv 1 \pmod{4}$?
   - $d \equiv 3 \pmod{4}$: Maxwell
   - $d \equiv 0, 2 \pmod{4}$: this work

2. Any hope of bijective proofs?
   - E.g., higher-dimensional Prüfer code, Joyal bijection, …
Thanks for listening!
Appendix A: The Weighted CMTTPV

Weighted Cellular Matrix-Tree Theorem, Pdet Version

Ingredients:

- \( S \) \text{ cell complex of dimension } \geq k
- \( \partial = \partial_k \)
- \( x = (x_i) \) \text{ variables indexing } (k-1)\text{-cells}
- \( X = \text{diag}(x) \)
- \( y = (y_i) \) \text{ variables indexing } k\text{-cells}
- \( Y = \text{diag}(y) \)

Formula:

\[
p\det(X^{1/2} \cdot \partial \cdot Z \cdot \partial^{tr} \cdot Y^{1/2}) = \tau_k(S; y) \tau_{k-1}(S; z^{-1}).
\]

Setting \( y_i = z_i = 1 \) recovers the unweighted formula.
Appendix B: A Little Linear Algebra

∂: matrix of rank $r$
$I, I'$: sets of $r$ rows
$J, J'$: sets of $r$ columns

**Useful Fact 1** ("The Minor Miracle")
$I$ and $J$ are a row basis and a column basis respectively if and only if $\det \partial_{I,J} \neq 0$.

**Useful Fact 2**

$$\det \partial_{I,J} \det \partial_{I',J'} = \det \partial_{I,J'} \det \partial_{I',J}.$$  

Important consequences for matrices that are (skew-)symmetric!
### Appendix C: Explicit Reorientation of Simplices

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