Amazing Patterns in the Game of Nim

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University of Kansas

Inspired by Math
October 25, 2016
Nim is a game for two players, taking turns. Start with a bunch of piles of “chips” of different colors.
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On your turn, you can remove as many chips as you like of any one color.
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\[
\begin{array}{c}
\includegraphics[width=\textwidth]{nim_game.png}
\end{array}
\]
Now it’s your opponent’s turn to pick a color and remove as many chips as she likes of that color (again, at least one).

The Game of Nim

Now it’s your opponent’s turn to pick a color and remove as many chips as she likes of that color (again, at least one).

Then it’s your turn again.
The Game of Nim

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The goal is to take the last chip. Whoever does that wins the game.
The Game of Nim

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Then it’s your turn again.

The goal is to take the last chip. Whoever does that wins the game.

Passing isn’t allowed (why not?)
A Sample Game

Amy wins!
A Sample Game

Amy wins!
A Sample Game

Amy

Bob

Amy wins!
A Sample Game

Amy
Bob
Amy
Bob
Amy

Amy wins!
A Sample Game

Amy

Bob

Amy

Bob

Amy

Bob

Amy wins!
A Sample Game

Amy

Bob

Amy

Bob

Amy

Bob

Amy

Bob

Amy

Bob

Amy

Bob

Amy wins!
A Sample Game

Amy

Bob

Amy

Bob

Amy wins!
A Sample Game

Amy  Bob

Amy  Bob

Amy  Bob

Amy  Bob

Amy wins!
A Sample Game

Amy wins!
What’s the Best Strategy?

First, let’s be more efficient about how we describe Nim positions. We don’t need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.
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```
\[ \text{\begin{tabular}{c}
| \hspace{1cm} | \hspace{1cm} | \hspace{1cm} | \hspace{1cm} |
|-----------------|
| \hspace{1cm} | \hspace{1cm} | \hspace{1cm} | \hspace{1cm} |
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| \hspace{1cm} | \hspace{1cm} | \hspace{1cm} | \hspace{1cm} |
|-----------------|
| \hspace{1cm} | \hspace{1cm} | \hspace{1cm} | \hspace{1cm} |
\end{tabular}} \quad = \quad \begin{pmatrix} 5 & 4 & 2 & 5 \end{pmatrix} \]
```
What’s the Best Strategy?

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In fact, we don’t need to distinguish between colors.
What’s the Best Strategy?

First, let’s be more efficient about how we describe Nim positions.

We don’t need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.

![Nim position diagram]

In fact, we don’t need to distinguish between colors.
What’s the Best Strategy?

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We don’t need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.

\[
\begin{array}{cccc}
\text{red} & \text{green} & \text{orange} & \text{blue} \\
3 & 2 & 1 & 4 \\
\end{array}
= \begin{array}{ccc}
5 & 4 & 2 & 5 \\
\end{array}
\]

In fact, we don’t need to distinguish between colors.

\[
\begin{array}{cccc}
\text{red} & \text{green} & \text{orange} & \text{blue} \\
3 & 2 & 1 & 4 \\
\end{array}
= \begin{array}{ccc}
4 & 5 & 5 & 2 \\
\end{array}
\]
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We don’t need to keep drawing pictures of colored dots: all we need to know is how many chips there are of each color.

In fact, we don’t need to distinguish between colors.
The game tree keeps track of all the possible moves.

(Maybe “tree” isn’t the best word, since the branches can meet up with each other... )
Game Trees

If you know the entire game tree, then you can calculate all possible variations and find the best move.
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Game Trees

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How do you analyze this thing?
If you know the entire game tree, then you can calculate all possible variations and find the best move.

How do you analyze this thing?
Game trees can get big.
## Amazing Pattern #1

What are these numbers anyway?

<table>
<thead>
<tr>
<th>Starting position</th>
<th>Size of game tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1 1</td>
<td>3</td>
</tr>
<tr>
<td>1 1 1</td>
<td>4</td>
</tr>
<tr>
<td>1 1 1 1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2 2</td>
<td>6</td>
</tr>
<tr>
<td>2 2 2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3 3</td>
<td>10</td>
</tr>
<tr>
<td>3 3 3</td>
<td>20</td>
</tr>
<tr>
<td>3 3 3 3</td>
<td>35</td>
</tr>
</tbody>
</table>
### Amazing Pattern #1

<table>
<thead>
<tr>
<th>Starting position</th>
<th>Size of game tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
Amazing Pattern #1

Starting position

```
0
1
2
3
4
```

Size of game tree

```
1
2
3
4
5
```
Amazing Pattern #1

Starting position

```
  0
  1
 11  2
 22  3
 33  4
```

Size of game tree

```
  1
  2
  3
  4
  5
```
Amazing Pattern #1

Starting position

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>11</td>
<td>22</td>
<td>33</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Size of game tree

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
Amazing Pattern #1

Starting position

0
0
11
22
33

Size of game tree

1
1
2
3
3
6
10
4
5
Amazing Pattern #1

**Starting position**

\[
\begin{array}{ccc}
0 & 1 & 0 \\
0 & 1 & 0 \\
11 & 2 & 0 \\
22 & 3 & 0 \\
33 & 4 & 0 \\
\end{array}
\]

**Size of game tree**

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 1 \\
3 & 3 & 1 \\
6 & 4 & 1 \\
10 & 5 & 1 \\
\end{array}
\]
Amazing Pattern #1

Starting position

<table>
<thead>
<tr>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>111</td>
<td>22</td>
</tr>
<tr>
<td>222</td>
<td>33</td>
</tr>
</tbody>
</table>

Size of game tree

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>
Amazing Pattern #1

Starting position

0
0 0
0 1 0
0 11 2 0
0 111 22 3 0
1111 222 33 4 0

Size of game tree

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
5 10 10 5 1
Amazing Pattern #1

Starting position

```
0
0 0
0 1 0
0 11 2 0
0 111 22 3 0
0 1111 222 33 4 0
```

Size of game tree

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
```
## Amazing Pattern #2

<table>
<thead>
<tr>
<th>Starting position</th>
<th>Size of game tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2 1</td>
<td>5</td>
</tr>
<tr>
<td>3 2 1</td>
<td>14</td>
</tr>
<tr>
<td>4 3 2 1</td>
<td>42</td>
</tr>
<tr>
<td>5 4 3 2 1</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>10 9 8 7 6 5 4 3 2 1</td>
<td>16796</td>
</tr>
</tbody>
</table>

(To find out what is so amazing about these numbers, come to Dr. Jennifer Wagner’s Inspired by Math talk this coming spring!)
Game Trees Can Get Really Big

For a game like chess, where each player might have 20 different available moves on his or her turn, the number of different things that could happen over the next 6 moves is

$$20 \times 20 \times 20 \times 20 \times 20 \times 20 = 20^6 = 64,000,000.$$  

For a game that lasts 30 moves (which is a very short game!), the number of possibilities would be

$$20^{30} = 1,073,741,824,000,000,000,000,000,000,000,000,000,000.$$  

Just making a list of all these games would require about a million million million Internets (give or take).\(^1\)

\(^1\)Source: http://www.sciencefocus.com/qa/how-many-terabytes-data-are-internet. This article was published in 2013. Maybe in 2016 it would only take a thousand million million Internets.
Trying to understand the game tree for Nim looks utterly hopeless.

But there is good news: You can play Nim perfectly without ever thinking about the game tree!

The key is to start small. Let’s look at some really simple games of Nim and work our way up to more complicated ones.

In all of these games, **Amy** will go first and **Bob** will go second.
How about the starting position 1?
How about the starting position 1?

Here Amy has only one move — and it’s a winning one.
How about the starting position 2?
How about the starting position 2?

Now Amy has two moves. One is bad and one is good. With best play, she can win.
How about the starting position 702?
How about the starting position 702?

Amy can win by taking all the chips. (There are also 701 bad moves that let Bob win, but all you need is one good move.) Any one-pile game is a win for Amy.
Simple Nim Game #0

How about the starting position 0?
How about the starting position 0?

This is a game that someone just won.
More precisely, it’s a game that Bob just won.
How about the starting position 0?

This is a game that someone just won.

More precisely, it’s a game that Bob just won.

This may look silly, but actually it’s a very important game. (After all, every game reaches this point eventually!)
What can we learn from the one-pile game?
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- If only one pile of chips is left, then the first player can win.
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- Every game will eventually get to the point where there is only one pile of chips left.
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- In other words, the winning strategy is to force your opponent to take the last chip of the second-to-last color.
What can we learn from the one-pile game?

- If only one pile of chips is left, then the first player can win.
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    **Your goal is to make sure it is your move when that happens.**

- In other words, the winning strategy is to force your opponent to take the last chip of the second-to-last color.

Let’s look at some two-pile games.
The game $1 \ 1$:

Now Amy has 2 moves... but they're both equivalent, and they're both losing moves.

Amy has no winning move, so $1 \ 1$ is a win for Bob.
The game $2 \ 1$: What should Amy do?
Two-Pile Nim: Example #2

The game $2 \ 1$:

The game $2 \ 1$:
The game $2 \ 1$:

2 1 is a win for Amy.
The game 2 2:

What should Amy do?
The game 2 2:

Taking two chips is a bad idea.

What should Amy do?
Two-Pile Nim: Example #3

The game 2 2:

What should Amy do?

Taking two chips is a bad idea.

But taking one chip is no better.
Two-Pile Nim: Example #3

The game 2 2:

What should Amy do?

Taking two chips is a bad idea.

But taking one chip is no better.

2 2 is a win for Bob.
A-Positions and B-Positions

In some positions, the first player can ensure a win with best play, no matter what the second player does. We’ll call these **A-positions**.

For example: 1, 2, 702, 2 1, ...

In some other positions, the roles are reversed — the second player can ensure a win with best play. We’ll call these **B-positions**.

For example: 1 1, 2 2, 0, ...

**Important Fact:**
Every Nim position is either an A-position or a B-position.

This may or may not be clear to you right now, but I promise to explain it soon!
Two-Pile Nim: More Examples

What about these positions?

```
A-position

B-position
```
Two-Pile Nim: More Examples

What about these positions?

Try them yourself — I’ll wait.
What about these positions?

A-position
What about these positions?

A-position

A-position

A-position
Two-Pile Nim: More Examples

What about these positions?

- A-position
- A-position
- B-position
Theorem: A Nim game with two piles is...

- a **B-position** (second-player win) if the piles have **the same size**, and
- an **A-position** (first-player win) if the piles have **different sizes**.

Proof:

- If the piles have the same size, then **Bob** can win with a “copycat” strategy, forcing **Amy** to be the first player to remove a pile entirely.
- On the other hand, if the piles have different sizes, then **Amy** can win by using her first move to equalize them, producing a B-position.

\[\square\]
## Summary: Two Or Fewer Piles

<table>
<thead>
<tr>
<th>Number of piles</th>
<th>Winner (assuming best play)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Bob</td>
</tr>
<tr>
<td>1</td>
<td>Amy</td>
</tr>
<tr>
<td>2 (equal)</td>
<td>Bob</td>
</tr>
<tr>
<td>2 (unequal)</td>
<td>Amy</td>
</tr>
</tbody>
</table>
Okay, what about three piles?
Theorem: Every Nim position with three piles, including two of the same size, is an A-position.

Proof: Amy can win by removing an entire pile, leaving two of the same size.
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Proof: Amy can win by removing an entire pile, leaving two of the same size.
What about three piles of all different sizes? For example: 3 2 1.
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Two unequal piles

Three piles, two equal

Two unequal piles

Three piles, two equal

Three piles, two equal

Two unequal piles
What about three piles of all different sizes? For example: 3 2 1.

3 2 1 is a B-position.
Three Unequal Piles Including a 1

- $3 \ 2 \ 1$ is a B-position.

Therefore $4 \ 2 \ 1$ is an A-position, since Amy can move to $3 \ 2 \ 1$.

So are $5 \ 2 \ 1$ and $6 \ 2 \ 1$ and $7 \ 2 \ 1$ and...

So are $4 \ 3 \ 1$ and $5 \ 3 \ 1$ and $6 \ 3 \ 1$ and...

$5 \ 4 \ 1$ is a win for Bob.

Therefore $6 \ 4 \ 1$ and $7 \ 4 \ 1$ and $8 \ 4 \ 1$ are A-positions.

And so are $6 \ 5 \ 1$ and $7 \ 5 \ 1$ and $8 \ 5 \ 1$, etc.

But $7 \ 6 \ 1$ is a B-position

So are $9 \ 8 \ 1$ and $11 \ 10 \ 1$ and...

The upshot:

If $x > y$, then $x \ 1 \ y$ is a B-position if and only if $y$ is even and $x = y + 1$. 
Three Unequal Piles Including a 1

- **3 2 1** is a B-position.
  - Therefore **4 2 1** is an A-position, since Amy can move to **3 2 1**.

- **5 4 1** is a win for Bob.
  - Therefore **6 4 1** and **7 4 1** and **8 4 1** are A-positions.
  - And so are **6 5 1** and **7 5 1** and **8 5 1**, etc.

- **7 6 1** is a B-position.
  - So are **9 8 1** and **11 10 1** and.

The upshot: If \( x > y \), then \( x y 1 \) is a B-position if and only if \( y \) is even and \( x = y+1 \).
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  - So are **4 3 1** and **5 3 1** and **6 3 1** and…

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Three Unequal Piles Including a 1

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  - Therefore **4 2 1** is an A-position, since Amy can move to **3 2 1**.
  - So are **5 2 1** and **6 2 1** and **7 2 1** and...
  - So are **4 3 1** and **5 3 1** and **6 3 1** and...

- **5 4 1** is a win for Bob.
Three Unequal Piles Including a 1

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  - Therefore **4 2 1** is an A-position, since Amy can move to **3 2 1**.
  - So are **5 2 1** and **6 2 1** and **7 2 1** and... 
  - So are **4 3 1** and **5 3 1** and **6 3 1** and... 

- **5 4 1** is a win for Bob.
  - Therefore **6 4 1** and **7 4 1** and **8 4 1** are A-positions.
Three Unequal Piles Including a 1

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  - Therefore **4 2 1** is an A-position, since Amy can move to **3 2 1**.
  - So are **5 2 1** and **6 2 1** and **7 2 1** and...
  - So are **4 3 1** and **5 3 1** and **6 3 1** and...

- **5 4 1** is a win for Bob.
  - Therefore **6 4 1** and **7 4 1** and **8 4 1** are A-positions.
  - And so are **6 5 1** and **7 5 1** and **8 5 1**, etc.
Three Unequal Piles Including a 1

- **3 2 1** is a B-position.
  - Therefore **4 2 1** is an A-position, since Amy can move to **3 2 1**.
  - So are **5 2 1** and **6 2 1** and **7 2 1** and...
  - So are **4 3 1** and **5 3 1** and **6 3 1** and...

- **5 4 1** is a win for Bob.
  - Therefore **6 4 1** and **7 4 1** and **8 4 1** are A-positions.
  - And so are **6 5 1** and **7 5 1** and **8 5 1**, etc.

- But **7 6 1** is a B-position
Three Unequal Piles Including a 1

- **3 2 1** is a B-position.
  - Therefore **4 2 1** is an A-position, since Amy can move to **3 2 1**.
  - So are **5 2 1** and **6 2 1** and **7 2 1** and... 
  - So are **4 3 1** and **5 3 1** and **6 3 1** and...

- **5 4 1** is a win for Bob.
  - Therefore **6 4 1** and **7 4 1** and **8 4 1** are A-positions.
  - And so are **6 5 1** and **7 5 1** and **8 5 1**, etc.

- But **7 6 1** is a B-position
- So are **9 8 1** and **11 10 1** and...

**The upshot:** If \( x > y \), then \( x \ y \ 1 \) is a B-position if and only if \( y \) is even and \( x = y + 1 \).
What we are doing is sorting Nim games into two types: **A-positions** and **B-positions**.

We are working our way up from simple games to more complex ones.

- If there is some way to move from the current position to a B-position, then the current position is an A-position. (Amy has a winning move.)

- Otherwise, the current position is a B-position. (Amy has nothing but losing moves.)

(Does this make more sense now?)
Here is what we know so far:

- **A-positions:** 1 pile; 2 unequal piles; 3 piles with at least two equal
- **B-positions:** 0; 2 equal piles

What about three unequal piles?
Most are **A-positions**, but here are some that are **B-positions**...

1 2 3  
2 4 6  
3 4 7  
4 8 12  
5 8 13  
6 8 14  
7 8 15  
8 16 24  
9 16 25  

1 4 5  
2 5 7  
3 5 6  
4 9 13  
5 9 12  
6 9 15  
7 9 14  
8 17 25  
9 17 24  

1 6 7  
2 8 10  
3 8 11  
4 10 14  
5 10 15  
6 10 12  
7 10 13  
8 18 26  
9 18 27  

1 8 9  
2 9 11  
3 9 10  
4 11 15  
5 11 14  
6 11 13  
7 11 12  
8 19 27  
9 19 26  

1 10 11  
2 10 12  
3 12 15  
4 16 20  
5 16 21  
6 16 22  
7 16 23  
8 20 28  
9 20 29  

1 12 13  
2 13 15  
3 13 14  
4 17 21  
5 17 20  
6 17 23  
7 17 22  
8 21 29  
9 21 28  

...
Reminder: Every number can be written in binary.

Decimal (base ten): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . . , 99, 100, 101, . . .
Binary (base two): 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, . . .

$$101011_{\text{bin}} = 2^5 + 2^3 + 2^1 + 2^0$$
$$= 32 + 8 + 2 + 1$$
$$= 43_{\text{dec}}$$
**Reminder:** Every number can be written in **binary**.

*Decimal* (base ten): 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . . , 99, 100, 101, . . .

*Binary* (base two): 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, . . .

\[
101011_{\text{bin}} = 2^5 + 2^3 + 2^1 + 2^0 \\
= 32 + 8 + 2 + 1 \\
= 43_{\text{dec}}
\]

“There are 10 kinds of people in the world. Those who can count in binary and those who can’t.”
Let’s look at some three-pile Nim games — but write the pile sizes in **binary** instead of decimal, and stack them on top of each other.

<table>
<thead>
<tr>
<th></th>
<th>112</th>
<th>122</th>
<th>123</th>
<th>124</th>
<th>125</th>
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</tr>
</tbody>
</table>

What’s the pattern?
Let’s look at some three-pile Nim games — but write the pile sizes in **binary** instead of decimal, and stack them on top of each other.

<table>
<thead>
<tr>
<th>112</th>
<th>122</th>
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What’s the pattern?
Three-Pile Nim and Binary Numbers

Let’s look at some three-pile Nim games — but write the pile sizes in **binary** instead of decimal, and stack them on top of each other.

<table>
<thead>
<tr>
<th></th>
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<td>2 2</td>
<td>2 2</td>
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</tr>
</tbody>
</table>

What’s the pattern? Look at the “Nimbers.”
Observation: In a B-position, all Nimbers are even.

Having made this observation, what do we do next?
Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
Observation: In a B-position, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).
Observation: In a B-position, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).
3. Figure out why it works.
**Observation:** In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).
3. Figure out why it works.

These steps are the same as those we would carry out in any other science (chemistry, physics, biology, . . .) . . . but in mathematics there’s an additional step:
Observation: In a **B-position**, all Nimbers are even.

Having made this observation, what do we do next?

1. Check that it works for other cases that we know about.
2. See whether it works in other cases (and refine it if necessary).
3. Figure out why it works.

These steps are the same as those we would carry out in any other science (chemistry, physics, biology, ...) ... but in mathematics there’s an additional step:

4. **Prove** that it **always** works!
Guess: In a B-position, all Nimbers are even.

Does this pattern work in the 2-pile game?
Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? **Sure!**
Guess: In a **B-position**, all Nimbers are even.

*Does this pattern work in the 2-pile game?* **Sure!**

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is $1$. 

Guess: In a **B-position**, all Nimbers are even.

Does this pattern work in the 2-pile game? **Sure!**
- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

Does this pattern work in the 1- and 0-pile games?
**Guess:** In a **B-position**, all Nimbers are even.

*Does this pattern work in the 2-pile game? *Sure!*

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

*Does this pattern work in the 1- and 0-pile games? *Sure!*
**Guess:** In a **B-position**, all Nimbers are even.

*Does this pattern work in the 2-pile game?** Sure!
- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

*Does this pattern work in the 1- and 0-pile games?** Sure!
- If there is one pile then at least one digit (possibly several) occurs once.
- If there are no piles then there aren’t any digits.
Does this pattern work in the 2-pile game? **Sure!**

- If the two piles are equal then every Nimber is certainly even.
- If they are different then there is some digit that occurs in one pile but not the other, so one of the Nimbers is 1.

Does this pattern work in the 1- and 0-pile games? **Sure!**

- If there is one pile then at least one digit (possibly several) occurs once.
- If there are no piles then there aren't any digits.

Does this pattern work in the 4-pile game?
Conjecture: In a **B-position**, all Nimbers are even.
**Conjecture:** In a **B-position**, all Nimbers are even.

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>2</th>
<th>1</th>
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</thead>
<tbody>
<tr>
<td>4</td>
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<td>5</td>
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<tr>
<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Bob can win by following a copycat strategy.

Who wins with best play?
Conjecture: In a **B-position**, all Nimbers are even.

<table>
<thead>
<tr>
<th></th>
<th>4</th>
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<th>1</th>
</tr>
</thead>
<tbody>
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<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Who wins with best play?

**Bob can win** by following a copycat strategy.
Conjecture: In a **B-position**, all Nimbers are even.

<table>
<thead>
<tr>
<th>4</th>
<th>2</th>
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<th>Who wins with best play?</th>
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</thead>
<tbody>
<tr>
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<tr>
<td>6</td>
<td>1</td>
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</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Who wins with best play?
**Conjecture:** In a B-position, all Nimbers are even.

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>2</th>
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</thead>
<tbody>
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<tr>
<td>5</td>
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<tr>
<td>6</td>
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</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>

Who wins with best play?

**Bob can win** (details left to the reader).
**Conjecture:** In a **B-position**, all Nimbers are even.

Who wins with best play?

<table>
<thead>
<tr>
<th></th>
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</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Bob can win** (details left to the reader).

**Idea:** Amy must always make some Nimber odd and Bob can always make them all even.
The Universal Foolproof Strategy For Winning At Nim

Make sure that after every move you make, all Nimbers are even.
Make sure that after every move you make, all Nimbers are even.
For example, suppose the starting position is 5 4 3 2 1.

\[
\begin{array}{ccc}
4 & 2 & 1 \\
5 & = & 1 \\
4 & = & 1 \\
3 & = & 0 \\
2 & = & 0 \\
1 & = & 0 \\
\end{array}
\]

\[\begin{array}{ccc}
2 & 2 & 3
\end{array}\]
For example, suppose the starting position is 5 4 3 2 1.

<table>
<thead>
<tr>
<th>4</th>
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<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
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<tr>
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<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

What move(s) will keep the 4-Nimber and the 2-Nimber even and change the 1-Nimber from odd to even?
For example, suppose the starting position is $5 \ 4 \ 3 \ 2 \ 1$.

<table>
<thead>
<tr>
<th>4</th>
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</thead>
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<tr>
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</tbody>
</table>

$2 \ 2 \ 3$

What move(s) will keep the 4-Nimber and the 2-Nimber even and change the 1-Nimber from odd to even?

There are two possibilities.
For example, suppose the starting position is 5 4 3 2 1.

<table>
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<tr>
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<tr>
<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>

What move(s) will keep the 4-Nimber and the 2-Nimber even and change the 1-Nimber from odd to even?

There are two possibilities.
The Universal Foolproof Strategy For Winning At Nim

**Theorem:** A Nim position is a **B-position** if every Nimber is even, and a **A-position** if at least one Nimber is odd.

**Proof:** I have to convince you of three things:

1. If every Nimber is even, then **every** move Amy can possibly make will turn some Nimber odd.
2. If at least one Nimber is odd, then Amy has **some** move that will turn all Nimbers even.

- For #1, every move changes one pile, therefore changes at least one Nimber from even to odd.
- For #2, look for a pile that contributes to the biggest Nimber.
Applying The Strategy

For example, if the starting position is \textbf{52 34 25 23 17 4}, then...

\begin{tabular}{lcccccc}

<table>
<thead>
<tr>
<th></th>
<th>32</th>
<th>16</th>
<th>8</th>
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</tbody>
</table>

\end{tabular}

\begin{tabular}{cccccc}

|   | 2 | 4 | 1 | 3 | 2 | 3 |

\end{tabular}

The biggest Nimber is 8. The only odd Nimbers are 8, 4, 1. The winning move is to change 0 1 ˙1 ˙0 0 ˙1 to 0 1 0 1 0 0.
For example, if the starting position is 52 34 25 23 17 4, then...

<table>
<thead>
<tr>
<th></th>
<th>32</th>
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<th>8</th>
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<th>1</th>
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<tr>
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<td>= 0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>= 0</td>
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<td>0</td>
<td>1</td>
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</table>

Biggest Nimber: 8
Applying The Strategy

For example, if the starting position is 52 34 25 23 17 4, then...

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</tbody>
</table>

Biggest Nimber: 8
All odd Nimbers: 8, 4, 1

Winning move: Change 0 1 0 1 0 0 to 0 1 0 1 0 0.
Applying The Strategy

For example, if the starting position is 52 34 25 23 17 4, then . . .

\[
\begin{array}{cccccc}
32 & 16 & 8 & 4 & 2 & 1 \\
52 & = & 1 & 1 & 0 & 1 & 0 & 0 \\
34 & = & 1 & 0 & 0 & 0 & 1 & 0 \\
25 & = & 0 & 1 & 1 & 0 & 0 & 1 \\
23 & = & 0 & 1 & 0 & 1 & 1 & 1 \\
17 & = & 0 & 1 & 0 & 0 & 0 & 1 \\
4 & = & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
& & 2 & 4 & 1 & 3 & 2 & 3 \\
\end{array}
\]

Biggest Nimber: 8
All odd Nimbers: 8, 4, 1

**Winning move:** Change 0 1 i 0 0 i to 0 1 0 1 0 0.
What Next?

What about **Misère Nim**? ("Misère" means that whoever takes the last chip **loses**.)

- The strategy is actually very similar to regular Nim until you get down to two small piles.

What about **other games**?

- The **Sprague-Grundy Theorem** says that many other games can be modeled using Nim!
- (Specifically: all **two-player, finite, impartial** games.)

- More complex games require more complex mathematics...
Thank you very much!

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