On the Chromatic Symmetric Function of a Tree

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Warning! Attention! ¡Cuidado!

Our FPSAC ’06 extended abstract has been superseded by stronger results.

Please refer to the article “On the Chromatic Symmetric Function of a Tree” by Jeremy Martin, Matthew Morin, and Jennifer Wagner (in preparation).
Chromatic Symmetric Functions

$G =$ finite simple graph
$V(G) =$ vertices
$E(G) =$ edges
$n = \#V(G)$
$x = \{x_1, x_2, \ldots \}$ = commuting indeterminates

**Coloring of** $G$: a function $\kappa : V(G) \to \mathbb{N}$ such that

$$vw \in E(G) \implies \kappa(v) \neq \kappa(w)$$

**Chromatic symmetric function of** $G$:

$$X_G = X_G(x) = \sum_{\text{colorings } \kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

(Stanley 1995)

- Symmetric in $x_1, x_2, \ldots$
- Homogeneous of degree $n$
- Stronger invariant than the chromatic polynomial
Examples

- \( G = K_n \) (complete graph on \( n \) vertices)
  \[ X_G = e_n = e_n(x) \]

- \( G = \overline{K_n} \) (\( n \) vertices, no edges)
  \[ X_G = p_1^n = (x_1 + x_2 + \cdots)^n \]

- \( G = P_3 \)
  \[ X_G = 24m_{1111} + 6m_{211} + 2m_{22} \]

- \( G = S_3 \)
  \[ X_G = 24m_{1111} + 6m_{211} + m_{31} \]
$X_G$ is not a complete invariant

Example (Stanley): The following two nonisomorphic graphs have the same chromatic symmetric function:

Open Question: If $T$ is a tree, does $X_T$ determine $T$ up to isomorphism?

- Yes for $n \leq 23$ (Tan 2006)
- Yes for certain special families of graphs (spiders, some caterpillars)
**Coefficients of** \(X_G\)

For \(A \subseteq E(G)\), let \(\lambda(A)\) be the partition of \(n\) whose parts are the sizes of the components of \(A\).

\[
A = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \\
\bullet & & \\
\end{array}
\quad \lambda(A) = (4, 2, 2, 1)
\]

For all graphs \(G\):

\[
X_G = \sum_{A \subseteq E(G)} (-1)^{\#A} p_{\lambda(A)}.
\]

For trees \(T\):

\[
X_T = \sum_{\lambda \vdash n} c_{\lambda}(T') p_{\lambda}
\]

where

\[
c_{\lambda} = c_{\lambda}(T') = (-1)^{n-\ell(\lambda)} \# \{ A \subseteq E(T) \mid \lambda(A) = \lambda \}.
\]
Elementary Graph Invariants from $X_G$

- $n = |V(G)|$ = degree of $X_G$

- $|E(G)| = c_2 = c_{211\ldots1}$

- # connected components $= \min \{ \ell(\lambda) \mid c_\lambda \neq 0 \}$

- # leaf edges $= c_{n-1}$

- If $G$ is a tree and $k > 1$, then
  number of subtrees of $G$ with $k$ vertices $= c_k$.

...
The Subtree and Connector Polynomials

For trees $\emptyset \neq S \subseteq T$, let $L(S) = \{\text{leaf edges of } S\}$.

**Subtree polynomial of $T$:**

$$S_T = S_T(q, r) = \sum_{\emptyset \neq S \subseteq T} q^{\#S} r^{\#L(S)}$$

For $\emptyset \neq A \subseteq T$, let $K(A)$ be the unique minimal subset of $E(T) - A$ such that $A \cup K(A)$ is a tree.

**Connector polynomial of $T$:**

$$K_T = K_T(x, y) = \sum_{\emptyset \neq A \subseteq T} x^{\#A} y^{\#K(A)}$$

**Proposition** (Chaudhary-Gordon, 1991) The subtree and connector polynomials can be recovered from each other.
**Theorem** (JLM-Morin-JDW, 2006)

The subtree and connector polynomials of a tree can be recovered from its chromatic symmetric function.

Specifically, let

\[
\psi(\lambda, a, b) = (-1)^{a+b} \left( \begin{array}{c} \ell - 1 \\ \ell - n + a + b \end{array} \right) \sum_{k=1}^{\ell} \binom{\lambda_k - 1}{a}
\]

Then

\[
K_T(x, y) = \sum_{a>0} \sum_{b>0} x^a y^b \sum_{\lambda \vdash n} \psi(\lambda, a, b) c_\lambda(T).
\]

Equivalently, define a symmetric function \( \Psi_n \) by

\[
\Psi_n(x, y) = \sum_{a>0} \sum_{b>0} x^a y^b \sum_{\lambda \vdash n} \psi(\lambda, a, b) \frac{p_\lambda}{z_\lambda}.
\]

Then

\[
K_T(x, y) = \langle \Psi_n(x, y), X_T \rangle
\]

where \( \langle \cdot, \cdot \rangle \) is the usual Hall scalar product on the space of symmetric functions.
Sketch of the Proof

The coefficient of $x^a y^b$ in $K_T(x, y)$ is

\[ \#\{A \subseteq T \mid \#A = a, \#K(A) = b\} \]

which (via manipulatorics) equals

\[ \sum_{\lambda \vdash n} (-1)^{a+b+n-\ell(\lambda)} \left( \frac{\ell(\lambda) - 1}{\ell(\lambda) - n + a + b} \right) \sum_{F \subseteq T} \alpha(F'). \]

(*)

where

\[ \alpha(F') = \#\{A \mid \#A = a, A \cup K(A) \subseteq F\}. \]

The key observation is that

\[ \alpha(F) = \sum_{k=1}^{\ell(\lambda)} \binom{\lambda_k - 1}{a} \]

(**)

This depends only on $\lambda(F')$, so (*) can be rewritten as a linear combination of the $c_{\lambda}(T)$. 
A Positivity Property of $\Psi_n$

Rewrite $\Psi_n$ in the basis of homogeneous symmetric functions $h_\mu$ as

$$
\Psi_n(x, y) = \sum_{i,j} \sum_{\mu|n} \xi(\mu, i, j)x^i y^j h_\mu
$$

where $\xi(\mu, i, j) \in \mathbb{Q}$.

**Conjecture:** Let $\varepsilon(\mu)$ be the number of parts of $\mu$ of even length. Then

$$
(-1)^{\varepsilon(\mu)}\xi(\mu, i, j) \geq 0
$$

for all partitions $\mu$ and integers $i, j$.

- Easy to verify for small $n$ (using, e.g., Stembridge’s SF package for Maple).
- In general $\xi(\mu, i, j) \notin \mathbb{Z}$, but it appears that $z_\mu \cdot \xi(\mu, i, j) \in \mathbb{Z}$. 
Consequences of the Main Theorem

1. The path and degree sequences of $T$, i.e., the numbers

$$\pi_i = \# \{ \text{paths in } T \text{ with } i \text{ edges} \}$$

and

$$\delta_j = \# \{ \text{vertices of } T \text{ of degree } j \}$$

can be recovered from its chromatic symmetric function.

2. Membership in certain families of trees (spiders, caterpillars, . . . ) can be deduced from $X_T$.

The subtree and connector polynomials do not suffice to distinguish trees with $n \geq 11$ (Eisenstat-Gordon, 2006).

So we still do not know whether the chromatic symmetric function is a complete invariant.
A Little Entomology

A *spider* is a tree with exactly one vertex of degree $\geq 3$ (the *torso*).

A *caterpillar* is a tree whose nonleaf vertices form a path (the *spine*).

**Theorem** (JLM-JDW)

Every spider can be reconstructed from its chromatic symmetric function.

(In fact, from its subtree polynomial; most of the path numbers are elementary symmetric functions of the leg sizes.)
Caterpillars are *not* distinguished by their subtree polynomials; in fact there exist infinitely many counterexamples (Eisenstat-Gordon, 2006), starting at $n = 11$.

**Theorem** (Morin)

If $T$ is a *symmetric* caterpillar (i.e., it has an automorphism reversing the spine) then it is distinguished by $X_T$.

**Theorem** (JLM-JDW-Morin)

If $T$ is a caterpillar in which every spine vertex has a different positive number of adjacent leaves, then it is distinguished by $X_T$. 
Further Questions

- Prove the skew-positivity of $\Psi_n(x, y)$, preferably by finding a combinatorial interpretation for $z_\mu \xi_\mu$.

- Are there other special classes of trees distinguished by their chromatic symmetric function (e.g., binary trees)?

- Does the Eisenstat-Gordon construction of nonisomorphic trees with the same subtree polynomial produce two trees with the same chromatic symmetric function?