Enumerating cellular colorings, orientations, tensions and flows

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The chromatic polynomial of a graph

\( G = (V, E) \): graph (loops, multiple edges OK) with arbitrary orientation
\( n = |V|, m = |E|, k \in \mathbb{N} \)

Proper \( k \)-coloring: \( f : V \to [k] \) with \( vw \in E \implies f(v) \neq f(w) \)

Chromatic polynomial \( \chi_G(k) = \# \text{ proper } k \text{-colorings of } G \)

- \( \chi_G(k) = \text{ polynomial in } k = k^n - mk^{n-1} + \cdots \)
- Deletion-contraction: \( \chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k) \)
- Specialization of Tutte polynomial
- Stanley reciprocity theorem: comb. interp. for \( \chi(-k) \)
Orient $G$ arbitrarily; $\partial = \text{signed incidence/boundary matrix}$

\[
\begin{bmatrix}
-1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 & 1 & -1 & 1
\end{bmatrix}
\]

Flow: $(f_e)_{e \in E}$ orthogonal to all rows of $\partial$
Tension: $(t_e)_{e \in E}$ orthogonal to all flows
Proper coloring: row vector $c = (c_v)_{v \in V}$ with $c \partial$ nowhere-zero

Flows/colorings/tensions can be modular (values in $\mathbb{Z}/k\mathbb{Z}$) or integral (values in $\{-k + 1, -k + 2, \ldots, k - 1\} \subseteq \mathbb{Z}$)
Flows and tensions

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Modular vs. integral

**Modular** $k$-flows/$k$-tensions
- Flows and tensions form $\mathbb{Z}$-modules [Tutte ’47]
- Counted by polynomials in $k$; specializations of Tutte poly
- Same for any abelian group of cardinality $k$

**Integral** $k$-flows/$k$-tensions
- Sign vectors correspond to orientations
- Counting functions are polynomials in $k$ [Kochol ’02]
- Lattice points in inside-out polytopes [Beck–Zaslavsky ’05]
- Reciprocity for flows [Breuer–Sanyal ’12]
Cell Complexes

Goal: Extend theory of colorings/cuts/flows from graphs to cell complexes.

\( X = d \)-dimensional cell complex
\( F = \) facets \((d\)-dimensional faces\)
\( R = \) ridges \(((d - 1)\)-dimensional faces\)

\( \partial = \) cellular boundary matrix \( \in \mathbb{Z}^{R \times F} \)
\( \partial^* = \) cellular coboundary matrix \( \in \mathbb{Z}^{F \times R} \)
\( \partial_k^* = \partial \otimes \mathbb{Z}/k\mathbb{Z} \)
Cellular colorings, flows and tensions

\( X = \text{pure CW complex} \quad F, R = \text{facets, ridges} \)
\( K = [-k + 1, k - 1] \subset \mathbb{Z} \quad \partial = \partial X \in \mathbb{Z}^{\left|R\right| \times |F|} \)

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Cellular orientations and compatibility

Definition
An orientation of $X$ is a sign vector $\varepsilon \in \{1, -1\}^F$.

An orientation $\varepsilon$ and tension/flow $x \in \mathbb{Z}^F$ are compatible if $\varepsilon_f x_f \geq 0$ for every $f$.

$\varepsilon$ is acyclic if it is not compatible with any nonzero flow.
$\varepsilon$ is totally cyclic if for every facet $f$, there is a $\varepsilon$-compatible flow $x$ with $x_f > 0$. 
Properties of the modular chromatic function $\chi^*_X(k)$

1. Deletion/contraction for facet/ridge pairs with degree 1
2. Closed formula:
   \[
   \chi^*_X(k) = \sum_{J \subseteq F} (-1)^{|J|} |\tilde{H}^d(X_J; \mathbb{Z}_k)| \; k^{n-|J|}
   \]
3. Quasipolynomial in $k$; bound on period
4. All $\partial J$ unimodular $\implies$ polynomial in $k$, T-G invariant

- Generalizes chromatic polynomial of a graph
- Comparable theorems for tension/flow polynomials (simplicial case: Beck–Kemper)
Integral coloring reciprocity

Theorem

- Acyclic orientations of $X$ $\leftrightarrow$ regions of hyperplane arrangement $\mathcal{H}_X$ with normals $= \text{columns of } \partial$
- $(-1)^n \chi_X(-2k - 1) = \# \text{ compatible pairs } (\varepsilon, c)$
- $c$ integral $k$-coloring, $\varepsilon$ orientation
- $|\chi_X(-1)| = \# \text{ acyclic orientations of } X$

Proof: count lattice points in inside-out polytope $(-1, 1)^n \setminus \mathcal{H}_X$; apply Ehrhart-Macdonald reciprocity

(Graph case: Stanley '73, Greene '77)
Integral tension reciprocity

Nowhere-zero integral $k$-tensions $=$ lattice points in interior of inside-out polytope

$$T = K^F \cap \text{Rowsp } \partial \setminus B$$

where $B = \text{Boolean arrangement of coordinate hyperplanes}$

**Theorem**

- Acyclic orientations of $X \leftrightarrow$ regions of $T$
- $|\tau_X(-2k - 1)| = \# \text{ compatible pairs } (\varepsilon, \psi)$:
  - $\psi$ integral $k$-tension, $\varepsilon$ orientation
- $|\tau_X(-1)| = \text{number of acyclic orientations}$

(Graph case: Chen ’10, Dall ’08)
Integral flow reciprocity

Nowhere-zero integral $k$-flows $=$ lattice points in interior of inside-out polytope

$$W = K^F \cap \ker \partial \setminus B$$

where $B =$ Boolean arrangement of coordinate hyperplanes

**Theorem**

- Totally cyclic orientations of $X \longleftrightarrow$ regions of $W$
- $|\varphi_X(-2k - 1)| = \# compatible pairs $(\varepsilon, w)$: $w$ integral $k$-flow, $\varepsilon$ orientation
- $|\varphi_X(-1)| = number of totally cyclic orientations

(Graph case: Beck–Zaslavsky '06)
Modular reciprocity

Modular reciprocity is trickier.

Geometrically: Modular flows/tensions correspond to lattice points in a “periodic inside-out polytope”

**Difficult part:** How do you associate an orientation (i.e. a sign vector) with a modular flow?

Idea: Breuer–Sanyal ’12 (modular flow reciprocity for graphs)
Related work: Chen–Stanley ’12
Modular flow reciprocity

**Theorem**

Let $X$ be a cell complex with no coloops. Then

$$|\varphi^*(X)(-k)| = \# \left\{ (\tilde{w}, \sigma) : \tilde{w} \text{ is a } \mathbb{Z}_k \text{-flow on } X \text{ and } \sigma : \text{zero}(\tilde{w}) \rightarrow \{-1, 1\} \text{ extends to a totally cyclic orientation} \right\}$$

**Corollary**

$$|\varphi^*_X(-1)| = \text{number of totally cyclic orientations}$$
Modular tension reciprocity

**Theorem**

Let $X$ be a cell complex with no loops. Then

$$|\tau^*_X(-k)| = \# \left\{ (\bar{t}, \sigma) : \bar{t} \text{ is a } \mathbb{Z}_k\text{-tension on } X \text{ and } \sigma : \text{zero}(\bar{t}) \to \{-1, 1\} \text{ extends to an acyclic orientation} \right\}$$

**Corollary**

$$|\tau^*_X(-1)| = \text{number of acyclic orientations}$$
Modular reciprocity: proof sketch

(Idea + graph case: Breuer–Sanyal 2012)

For $k > 0$, interpret $\varphi^*_X(k)$ as sum of Ehrhart functions of disjoint union of components of $(-k, k)|^F|

\bar{x} \in (\mathbb{Z}_k)^F$ is a flow $\iff$ some (= any) lift $x \in \mathbb{Z}^F$ has $\partial x \in (k\mathbb{Z})^R

b \in \mathbb{Z}^R \rightsquigarrow P^\circ(b) = \{ w \in (0, 1)^F : \partial w = b \}

\varphi^*_X(k) = \sum_b \text{Ehr}(P^\circ_b, k)

Then apply Ehrhart-Macdonald reciprocity.
Modular reciprocity: proof sketch

Example: \( \partial = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \)

\( P^\circ(0, 0) = \text{point } (0, 0) \quad P^\circ(1, 2) = \text{line segment } (1, 0) \text{ to } (0, \frac{1}{2}) \)

\( P^\circ(3, 6) = \text{point } (1, 1) \quad P^\circ(2, 4) = \text{line segment } (1, \frac{1}{2}) \text{ to } (0, 1) \)

\( \varphi_X^*(k) = \text{number of interior lattice points in union of } k^{th} \text{ dilates} \)

\( |\varphi_X^*(-k)| = \text{number of lattice points in closed union of } k^{th} \text{ dilates} \)
Modular reciprocity: proof sketch

- Lattice points on boundaries of $P(b)$’s have coordinates 0 mod $k$, i.e., somewhere-zero modular flows (which may admit more than one totally cyclic orientation)

- For bijection between these lattice points and $(\bar{\nu}, \sigma)$, sign = choice of whether to lift 0 mod $k$ to 0 or $k \in \mathbb{Z}$ (requires integral reciprocity!)
Further Directions

1. Is there a non-TU cell complex $X$ whose modular chromatic function $\chi^*_X(k)$ is polynomial?

   Breuer–Sanyal: used KRS to interpret values of Tutte polynomial of a graph at positive integers (a la Reiner ’99).
   **Generalize to cell complexes whose tension and flow functions are polynomials?**

3. **Hopf algebra** point of view: chromatic polynomial = combinatorial Hopf morphism from graphs to polynomials;
   reciprocity = inversion of characters